Unique and Multiple Equilibria in a Macroeconomic Model with Environment: Stability Analysis and Transitional Dynamics

Hafsa Tanveer

Supervised by:

Dr. Azam Chaudhry and Dr. Rehana Naz

A thesis submitted to the Lahore School of Economics in partial fulfilment of

the requirements for the degree in M.Phil. in Economics

DECLARATION

I declare that this is my own, unaided work. It is being submitted for the Degree of M.Phil. in Economics to the Lahore School of Economics, Pakistan. It has not been submitted before for any degree or examination to any other University.

Hafsa Tanveer

ACKNOWLEDGEMENTS

I am deeply grateful to my supervisors, Dr. Azam Amjad Chaudhry and Dr. Rehana Naz, for their encouragement, guidance, and motivation, without which this research would not have been possible. Their valuable feedback has helped me rise to the challenge of completing this thesis.
I would also like to acknowledge the feedback I received from Dr. Theresa Thompson Chaudhry, associate professor of economics, and Dr. Naved Hamid director of the Center for Research in Economics and Business (CREB) for their valuable comments. I am indebted to the Economics Department and CREB at the Lahore School of Economics who have been an immense support.

Finally, I am grateful to my family and friends for their immense support, encouragement, and motivation during this time.

Contents

List of Tables

List of Figures

List of Symbols

C(t): Consumption

E(t): Environmental quality

K(t): Physical capital

Y(t): Output in the economy

α: Weight assigned to physical capital

γ: Weight assigned to environmental quality

σ: Inverse of intertemporal elasticity of substituion

ρ: Discount factor

τ: income tax

θ: quantity of emissions

T: Abatement expenditures

Abstract

This thesis develops and analyzes an economic growth model with environmental quality factor in the production and utility function. We first assume that the capital share α equals to the inverse of intertemporal elasticity of substitution σ and solve our model for the balanced growth path. Unique low growth equilibrium is attained when environmental quality is given less weight in the utility function. The multiple equilibria exist if environmental quality is given greater weight in the utility function. These results hold even when we relax the assumption $\alpha = \sigma$ and consider fairly general values of capital share and intertemporal elasticity of substitution. We study the stability and transitional dynamics of model. We conclude that an economy in which the environmental quality is given relatively less importance by the agents will be caught in low growth, high consumption poverty traps which is the case for developing countries whereas the economies can potentially reach a relatively low consumption, high growth steady state if they give greater importance to environmental quality.

Introduction

Economic growth is directly related to the level of output and welfare of any country but the existence of positive or negative externalities act as catalysts in increasing or decreasing this growth. Positive externalities play a prime role in enhancing growth while negative externalities can retard economic growth. Positive externalities like health capital, better public infrastructure, educational opportunities while negative externalities such as congestion or low environmental quality have a negative impact on health, infrastructure, education and can cancel out the positive impact on economic growth because these externalities are not realized by private agents and lead to coordination failures (Gupta and Barman, 2010).

In this era of development and growth the major issue in front of the world economists is to achieve growth without permanently damaging the environment. The destruction of the environment due to climate change is not only a negative influence on world output but also an irreparable loss to the people of developing countries in particular (Beckerman, 1992).

In the developed world, concern regarding environmental quality grew in the early 1960's. For the developing world saving the environment was less important than the economic growth. Globally, expanding industries are damaging the environment but are also helping in reducing poverty as well as in increasing standards of living across the globe.

A clash of interests exists between different groups within a society and among rich and poor nations as well when it comes to the welfare and benefits of economic growth. It can be viewed as a trade-off between better environmental quality and higher economic growth. According to Gradus and Smulders (1993) concern about a clean environment will crowd investment and lowers endogenous growth rate. Because pollution is a byproduct of the physical capital, and environmental quality can be improved if some part of output is devoted to abatement activates but this will adversely affect the long run growth equilibrium. Rising production and consumption entail greater extraction of resources due to higher demand for large amounts of energy and inputs and also greater quantities of waste products (Ehrlich and Holdren, 1971). Byrne (1997) also studied the tradeoff between economic growth and environment. According to him a policy that restricts economic growth to zero, in anticipation of protecting the quality of environment will actually worked in the opposite direction. As in such an economy the growth rate of pollution will rise because in this case technology advances will also be restrictive and less advanced technology will yield higher emissions.

Furthermore the extraction of natural resources beyond a certain point will put economic growth in jeopardy (Panayotou, 2000). In short, economic growth and environmental degradation go hand in hand until a certain point. Some believe that in the case of developing countries economic growth is at the expense of social welfare of people because there is scarcity of drinking water, improper sanitation facilities and air pollution. Others believe that attempts to improve environmental quality will accelerate economic growth. They support the notion that as income in a country rises, their demand for goods that require material inputs decreases so their demand for services increases which will lead to a better environmental quality. The basic idea behind this approach is taken from Kuznets's inverted U shape curve which explains the relationship between income inequality and growth in a country. Kuznet proposed that income inequality rises with economic growth but will decrease when there is growth beyond a certain level(shown in appendix A as well) (Kuznets 1965, 1966). Similarly in the case of Environment and growth, people believe that there is an Environmental Kuznets's Curve (EKC) that states that when a country achieves a considerable amount of capital only then it can allocate a sufficient amount of capital to abatement activities which is the reason for an inverted U-shaped relation between economic growth and environmental quality. This idea is accepted widely by economists that if a country wants to improve environmental quality then more attention should be given to growth which will result in an overall increase of income per capita (Stern, Common and Barbier, 1996).

Authors like Yanqing and Mingsheng (2012) studied a 3E model on energy consumption, environmental pollution and economic growth by using panel data for 30 Chinese provinces from 2001 to 2008 and they found that the contribution of physical capital in economic growth was over 50 percent and was the chief component in output while labor force contributed to economic growth around 25 percent in China's output expansion.

Research regarding the relationship between environment and growth is based on two approaches. The first one is the ad hoc approach of empirical estimation and the second is the theoretical approach. By empirical approach the EKC is plotted for countries and the relationship between income and economic growth is observed. The relationship between income and environmental quality can be of different signs: it can either be positive or negative dependent upon a country's development path (Grossman and Krueger, 1995). Or it can be a relationship between GNP per capita and the pollutants (such as sulphur dioxide (SO_2), carbon dioxide (CO_2)). The empirical findings for different countries support three ideas regarding environmental quality. First is the evidence that environmental quality has improved in the developed world over the span of 30 years. This is mainly because of the decline in emissions. But this is not the case for fast growing developing countries.

In a recent survey, the WHO (2014) analyzed $1,600$ cities around the world. According to it Delhi (India) has the lowest environmental quality (in terms of air quality) mainly because of industries and vehicles while Patna, Gwailor and Raipur are other cities that are in top four with the lowest environmental air quality. Overall, half of the twenty cities with low environmental quality are in India. Also, unfortunately Karachi (Pakistan) is ranked fifth among cities with the lowest air quality. So the real dilemma is that these third-world countries are not realizing the situation at hand and are taking no control measures to improve the environmental quality.

Second, research finds that improvements in environmental quality is due to reductions in costs of abatement due to technological effects. Third, some researchers say that when income rises, environmental quality also improve after a threshold point. But this path is different for rich and poor countries because at any given income level poor countries experience lower environmental quality as compared to an initially rich country but once abatement activity and environmental regulations are imposed in poor countries environmental quality starts converging.

In theoretical approach which involves building of a macroeconomic model and solving an endogenous growth model, balanced growth path or equilibrium stage and overlapping generations models (Panayotou, 2000).

Recently economists are studying the importance of a broad variety of factors in the economic growth of a country beyond simple capital and labour. Instead of adding new variables, researchers are trying to find local dynamic properties of endogenous growth models. In particular they are analyzing their transitional paths and observe their convergence towards steady states. Hosoya (2012) investigates the local dynamics of an endogenous growth model with a health factor. Similarly, Gupta and Barman (2010) proposed a model of endogenous growth with a special focus on the role of health capital, public infrastructure and environmental pollution. Dinda (2005) combines the stock of capital, pollution, stock of environment (natural capital) in an endogenous growth model. Models of Greiner (2005) and Economides and Philippopoulos

(2008) introduce environmental quality in endogenous growth models with public infrastructure expenditure.

Gupta and Barman (2010) studied the impact of environmental quality on the health and infrastructural facilities. They found a direct link between health, capital of workers, infrastructural facilities and environmental quality. They suggested that the government should allocate its budget between infrastructural development, public health and abatement activities because low environmental quality will have a negative impact on economic growth and on public health capital as well. Most of the existing literature on environmental quality has focused on the amenity value attached to it. In most papers they didn't take environmental quality as a productive input. For Example, Forster (1973) took physical capital as the only input in the production process. Aznar-Márquez and Ruiz-Tamarit (2005) built an endogenous growth model in which, renewable resource is part of the production process along with physical capital. They have also analyzed short and long growth equilibrium as well as convergence towards the steady growth rate. But in this paper we are using environmental quality as an input in production process.

Our model also revolves around an endogenous growth model but with environmental quality, we will examine its transitional and local dynamic properties.In most of the economic growth models, transitional dynamics have been studied by five different approaches, namely phase diagrams (e.g. Romer (1986)), local stability analysis (e.g. Benhabib and Perli (1993)), numerical methods (Mulligan (1993)), the explicit dynamic methods (e.g. Xie (1994)) and closed form solutions (e.g. Boucekkine and Ruiz-Tamarit (2008); Chilarescu (2011); Naz et al (2014)). The approach adopted in our paper is of local stability analysis (Benhabib and Perli, 1993), and authors like Hosoya (2012), Dinda (2005) and Park (2004) also adopted the approach of local stability analysis to explain transitional dynamics and stable growth paths. We will use eigenvalue or eigenvectors in order to solve for the stable growth paths and if at least one eigenvalue is negative or stable, the balanced growth path is absolutely stable.

In order to study the evolution of economies in the long run many researchers are examining the properties of equilibrium paths, local indeterminacy and transitional dynamics in neoclassical and endogenous growth models. Based on this avenue of research the core objective of this paper is to explain stability and transitional (short run growth) dynamics around balanced growth path (long run growth) with an environmental quality factor. The balanced path is a subject of importance because at this point all economic variables of interest grow at the same rate. (Gasper et al 2014).

The main focus of the paper is to explain local dynamics around each balance growth path equilibrium when there are multiple long run equilibria so that we could predict the future states of the economies based on their structural characteristics. We may be able to predict the transition of an economy from one point to another that whether it will follow the trajectories and reach its equilibrium point or it will deviate from its balanced growth path based on the choice of their control variable. It will provide an insight on how economies that start from the same initial values of the state variables (economies with identical technologies and preferences) may follow rather different trajectories because of the preferences of the initial values of the jumping variables (expectations) made by the economic agents(Antoci, Galeotti, and Russu, 2014) and Peréz (2007) also worked on local and global indeterminacy.

In this context, expectations about the environment and government policies play a crucial role in explaining the dynamics and the evolution of the economy towards the long run growth. We can illustrate that if the initial value of the state variable (which in our case is physical capital K (t)) is close enough to the equilibrium values, the transition is dependent upon the choice of the jumping variable (consumption) so there exists a continuation of trajectories that is followed by the economies during their evolution periods (transition state) to reach their steady states or equilibrium points (Antoci, Galeotti, and Russu, 2014).

In our endogenous growth model with environment, unique and multiple equilibria exist. Multiple equilibria include high growth equilibria or growth miracles and low growth equilibria or poverty traps. The existence of multiple equilibria implies that for same levels of environmental quality the relationship between environmental quality and growth is non-monotonic. Thus we can have countries with identical endowments of physical capital experiencing two different growth regimes due to difference in their preferences and expectations in the economy. Similarly, Alonso-Carrera and Freire-Séren (2004), Mino (2004), Chen (2007), built an endogenous growth model in which they have explained the cases of countries with identical initial conditions of departure have experience divergence in long run growth paths and multiple equilibria.

Hosoya (2012) explains existence of multiple equilibria (i.e., high- and low-growth equilibria) with the same dynamic properties of health but due to difference in private agents expectations regarding public health infrastructural expenditure it resulted in two growth regimes. In high growth case agents expect better health expenditures and will increase their physical capital which results in high income and higher tax generation. This will translate into better health infrastructure and higher growth. However, in lower growth, the case is reversed and thus country will get caught in low growth equilibrium. Gasper et al (2014) also explained that multiple equilibria are expected to provide an important theoretical benchmark when examining the growth and development of a country, including growth miracles and poverty traps.

This paper will contribute to the literature in multiple ways: First, we will develop a macroeconomic model which incorporates the environmental quality factor in the production function and utility function. The household maximizes the utility function subject to the constraint of stock of capital and exogenously determined environmental quality. The law of motion for environmental quality satisfies both loops of the EKC. Second, we look at a dynamical system which incorporates capital and environmental quality into a neoclassical model and along the balanced growth path we will investigate the possibilities of unique and multiple equilibria. Third,we will explain stability and transitional (short run growth) dynamics around balanced growth path (long run growth). Fourth, implications of the model for developing countries because environmental quality is not high on their agenda but if environment is part of the production function then economic growth is not possible without improving environmental quality.

Chapter 1

Macroeconomic Model with Environment

In this chapter we explicate the definition of environmental quality and its effect on economic growth. For this reason we have explained the framework of the model and have shown channels through which environmental quality is influencing growth. Also, we have maximized an agents utility with respect to physical capital and environmental quality constraint.

1.1 Framework

There is a strong interaction between economic growth and environmental quality due to environment economy links and responses. For years it was considered that economic growth is constrained by limited natural resources but recently economists have realized that growth is not only limited by finite resources. Instead a new factor can also limit growth and it's nature's limited ability to absorb waste materials. The environment possesses certain characteristics such as it serves the purpose of a ”sink” in which wastes such as harmful air, water, solid pollutants and toxic chemicals can be dumped. Second, it is a source of input for production. Third, would be that clean environment also enhances productivity of the factors of productions and fourth there is an amenity value attached to better environmental quality. In the economic sphere of an economy, production and allocation is determined by the forces of demand (preferences of economic agents), supply (factors of production) and market institutions (like government when induced taxes or provide subsidies)(Smulders, 1999).

So in order to maximize the utility of an agent with respect to our budget constraint, we will build a current value hamiltonian function because it will give us the time path of consumption and not just consumption at one point only. Since time is a continuous variable, the physical capital constraint can not be a static function because it will exhibit the changes in capital at each point in time so $\dot{K}$ is a differential equation of time t.

1.1.1 Production and utility functions

The representative agent has a standard Cobb-Douglas production function in which we are using two inputs physical capital and environmental quality for the production of output $Y(t)$:

$$Y(t) = K(t)^{\alpha} E(t)^{1-\alpha}, 0 < \alpha < 1, \tag{1.1}$$

where $Y(t)$ is the aggregate output in the economy and we are assuming that Y(t) is the only type of good being produced in the economy, $K(t)$ is the amount of physical capital and $E(t)$ is the environmental quality (representing the stock of renewable natural resources: land, air water, flora and fauna) that may rise or fall overtime dependent upon the use of environmental resource. By stock variable we mean that environmental factor is not constrained by time. Environmental quality can be measured at any point in time as it is assumed as an accumulable input (Gupta and Barman, 2010).

In the production function α is the weight assigned to the physical capital or it is also known as the output elasticity of the production function as production function explains the relationship of output to input. α will measure the responsiveness of change in physical capital or environmental quality to output, keeping all other variables equal. Weight assigned to the physical capital (α) is a constant parameter and its value is dependent upon the type of technology used in the economy. It can either be a capital augmenting technology or labour augmenting technology. In the given production function the environmental quality $E(t)$ also captures the productivity effect because stock of environment is also used as an input in the production process. $E(t)$ is not the choice variable for the agents as it is assumed that the individual agents in the economy will view the stock of environment as given for their production function (Dinda, 2005) and (Boucekkine et al, 2014). It can also be considered as the social overhead capital which makes environment an exogenous factor to the agents so consequently it will be government's responsibility to provide better environmental quality.
The utility of a representative agent is of the form:

$$U(C,E) = \frac{(CE^{\gamma})^{1-\sigma} - 1}{1-\sigma} e^{-\rho t}, \tag{1.2}$$

where C and E are the consumption and environmental quality respectively and γ is the weight of environmental quality (E) in the utility function and σ is the inverse of intertemporal elasticity of substitution. The utility function is nonseparable in consumption and environmental quality. Authors like Agénor (2008, 2010) considered health as an argument in utility function and Greiner (2005) introduced pollution as a factor in the utility function.

1.1.2 Law of motion for physical capital

Assume that the depreciation rate of physical capital is zero then the standard law of motion for physical capital stock is

$$\dot{K} = I_K, \tag{1.3}$$

where I_K is investment on physical capital. The after tax production (income) allocated to investment and production goes to the representative agent of economy and thus

$$(1 - \tau)Y = I_K + C, \tag{1.4}$$

where τ is a flat tax rate on output imposed by government. The resource constraint is

$$Y = C + I_K + I_E, \tag{1.5}$$

where I_E denotes investment on environment quality and is given by

$$I_E = \tau Y, \ 0 < \tau < 1. \tag{1.6}$$

The equation (1.3) for physical capital with the help of equations (1.5) and (1.6) takes following form:

$$\dot{K} = (1 - \tau)Y - C. \tag{1.7}$$

The final form of evolution equation for physical capital is

$$\dot{K} = (1 - \tau)K^{\alpha}E^{1-\alpha} - C, \ 0 < \tau < 1, \ K(0) = K_0. \tag{1.8}$$

1.1.3 Law of motion for environment quality

In the decentralized economy E is an exogenous variable; thus agent maximizes own utility ignoring the effect of the environment factor. For this reason a joint concavity condition on C and E is not required here. The environment factor evolves as follows:

$$\dot{E} = TY - \theta Y,\ 0 < T < \tau < 1,\ 0 < \theta < 1 \tag{1.9}$$

(Gupta and Barman, 2010) where TY is the abatement expenditure made by the government to improve environmental quality and T is known as abatement expenditure rate defined as a ratio of abatement expenditure to national income, θ is emission-output coefficient or the level of emissions. We are assuming emissions to be a flow variable because we can measure emissions over an interval of time. And emissions are assumed to be directly related to the production process of a manufactured goods in an economy (Gupta and Barman, 2010). Emissions are mainly affected by scale of production means emissions rise and fall with the scale of the economic activity and composition of the industrial output.

The changes in environmental quality depend upon the resource use and emissions but this process of degradation takes place gradually over time. In this constraint environmental quality is being referred to as a single index which is not differentiated by different natural resources and other state of environment indicators. The environment constraint shows how the quality of environment changes over time due to waste materials such as emissions (θ) and costs allocated to reduce emissions or waste products which are by products of production process. Environmental quality can also improve over time depending upon the abatement activities in an economy. When environment's ability to absorb waste is exhausted or exceeded by the level of emissions then environmental quality falls which in turn can also cause growth limitations because high abatement costs may be required to improve environmental quality which consequently results in lower rate of return to investment (Brock and Taylor, 2004). Technological progress in abatement creates a scale effect which brings emissions downwards while technological progress in production of goods often increases level of emissions (Brock and Taylor, 2004). But in our current model we are assuming a balanced budget which means government's revenue earned through income taxation is all spent on abatement expenditure and all agents in the economy derive equal utility from the government expenditure.

The government is spending a significant part of the budgetary revenue to spend on abatement activities so even if environmental quality cannot be improved at least it can be sustained. The part played by abatement expenditure is essential to improve environment and reduce pollution in production side, however this expenditure can only be made when the economy has achieved a certain level of physical capital (Dinda, 2005).

1.1.4 The representative agent problem

In our model we are assuming finite number of agents who will live for an infinite time horizon. All these agents are completely identical and all the variables in the model are considered continuous and differentiable functions of time t. We are considering a non-separable utility function in terms of consumption and environmental quality. Another feature of the model is that environmental quality is a part of production function as well as the utility function. The agent lives from period 0 until forever and discounts the future at rate $\rho > 0$ The dynamic optimization problem is

$$Max \quad \int_0^\infty \frac{(CE^\gamma)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \ \gamma \geq 0, \sigma > 0, \sigma \neq 1, \rho > 0, \tag{1.10}$$

subject to constraint (1.14). We are considering a closed decentralized economy in which decision making is distributed among agents or is localized within production units where τ is proportional income tax and ρ is the discount factor which measures the patience level of an agent, it actually shows how patient individuals are in an economy. While σ denotes the inverse of intertemporal elasticity of substitution which measures willingness of agents to substitute present consumption for future consumption. As σ approaches 0, it would imply that households are indifferent between consumption today and consumption tomorrow. The lower the value of sigma, higher will be the elasticity of substitution which means agents are putting a higher value on future consumption as ($\sigma = \frac{1}{\varepsilon}$), where ε is the elasticity of substitution. We are taking labor force $L(t) = 1$ therefore all variables are expressed in per capita amount. It exhibits normalized population growth in which we are assuming that population is not growing.

On the empirical strand, literature supports that household preferences towards better environmental quality vary with the income levels such that at low income levels the demand for environmental quality is ranked lower than demand for consumption of goods and services but when the case is reversed demand for better environmental quality is outweighed by the demand of consumption goods as individuals can not consume beyond a point and with each additional unit of consumption, utility to consume falls.

1.2 Solving the model around the equilibrium

The current value Hamiltonian function for this model is

$$H = \frac{(CE^{\gamma})^{1-\sigma} - 1}{1-\sigma} + \lambda[(1-\tau)K^{\alpha}E^{1-\alpha} - C] \tag{1.11}$$

where $C(t)$ is the control variable because agents can exercise control over consumption today and consumption tomorrow, $K(t)$ is state variable that can not be controlled completely but nevertheless is affected by what one chooses as control variable $\lambda(t)$ is the costate variable. For example, the amount of capital an agent will have tomorrow is dependent upon the amount one consumes today. So physical capital $K(t)$ is determined by the path of the consumption $C(t)$. The necessary first order conditions(See Appendix B) for optimal control are

$$\frac{\partial H}{\partial C} = (CE^{\gamma})^{-\sigma}(E^{\gamma}) - \lambda = 0 \tag{1.12}$$

$$\lambda = \frac{(E^{\gamma(1-\sigma)})}{(C^{\sigma})} \tag{1.13}$$

This equation shows that shadow price of physical capital is directly related to environmental quality and inversely related to consumption.

$$\dot{K} = (1-\tau)K^{\alpha}E^{1-\alpha} - C, \ , 0 < \tau < 1, \tag{1.14}$$

$$\dot{\lambda} = -\lambda\alpha K^{\alpha-1}E^{1-\alpha}(1-\tau) + \rho\lambda. \tag{1.15}$$

The transversality condition is

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)K(t) = 0. \tag{1.16}$$

The utility function is concave in consumption C. The environment factor E is exogenous variable so we do not require a joint concavity condition of utility function on C and E. The constraint given in (1.14) is jointly concave in K and C. The conditions of the Mangasarian (1966) sufficiency theorem are satisfied for our model and thus first order conditions are in fact sufficient.

Differentiating equation (1.13) with respect to time t, the growth rate of consumption is

$$\frac{\dot{C}}{C} = \frac{\gamma(1-\sigma)}{\sigma}\frac{\dot{E}}{E} - \frac{\dot{\lambda}}{\lambda\sigma} \tag{1.17}$$

is dependent upon growth rate of environmental quality and the weight assigned to environmental quality (γ), inverse of intertemporal elasticity of substitution and on growth rate of shadow price of physical capital. A lower value of inverse of intertemporal elasticity of substitution σ means that people are willing to substitute present consumption for future which will increase the growth rate of consumption. When agents give a higher weight to the environmental quality γ, higher will be the growth rate of consumption. Substituting the values of $\frac{\dot{E}}{E}$ and $\frac{\dot{\lambda}}{\lambda}$ in the above equation, we have

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}[\gamma(1-\sigma)(T-\theta)(\frac{K}{E})^{\alpha} + (\alpha)(\frac{K}{E})^{\alpha-1}(1-\tau) - \rho]. \tag{1.18}$$

The differential equations representing the dynamics of this model can be summarized as follows:

$$\frac{\dot{K}}{K} = (1-\tau)K^{\alpha-1}E^{1-\alpha} - \frac{C}{K} \tag{1.19}$$

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}[\gamma(1-\sigma)(T-\theta)(\frac{K}{E})^{\alpha} + (\alpha)(\frac{K}{E})^{\alpha-1}(1-\tau) - \rho]. \tag{1.20}$$

$$\frac{\dot{E}}{E} = (T-\theta)(\frac{K}{E})^{\alpha} \tag{1.21}$$

The growth of the environmental quality is dependent upon the gap between abatement and emissions. Higher abatement costs will increase the growth of the environmental quality while higher emissions will adversely affect the environment.

Introducing two new variables, $X = C/K$ (control variable), and $Z = K/E($ state variable) the system of three differential equations (1.19)-(1.21) reduces to a system of following two differential equations in terms of X and Z

$$\frac{\dot{X}}{X} = \frac{1}{\sigma}[\gamma(1-\sigma)(T-\theta)Z^{\alpha} + \alpha(1-\tau)Z^{\alpha-1} - \rho] - (1-\tau)Z^{\alpha-1} + X \tag{1.22}$$

$$\frac{\dot{Z}}{Z} = (1-\tau)Z^{\alpha-1} - X - (T-\theta)Z^{\alpha} \tag{1.23}$$

One should also note that $X(0) = \frac{C(0)}{K(0)}$ is not predetermined since $C(0)$ is the jump variable.

1.2.1 The decentralized equilibrium

We now proceed to characterize the decentralized equilibrium and balanced growth path (BGP). The decentralized equilibrium is defined as follows:

Definition: In the decentralized equilibrium, C, K, E and Y grow at the same constant growth rate and thus X and Z are constants. Accordingly, our notations for each steady state are as follows: $g = \frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{E}}{E} = \frac{\dot{Y}}{Y}, X = X^*, Z = Z^*$.

We exclude the trivial solutions $X^* = 0,\ Z^* = 0$ and obtain the following stationary values for X^* and Z^*:

$$Z^* = \left(\frac{g}{T-\theta}\right)^{\frac{1}{\alpha}}, \tag{1.24}$$

$$X^* = (1-\tau)\left(\frac{g}{T-\theta}\right)^{\frac{\alpha-1}{\alpha}} - g, \tag{1.25}$$

where growth rate g satisfies

$$\alpha(1-\tau)\left(\frac{g}{T-\theta}\right)^{\frac{\alpha-1}{\alpha}} = [\sigma-(1-\sigma)\gamma]g+\rho. \tag{1.26}$$

The stationary values of variables Z^* and X^* (see Appendix C) in terms of the growth rate, g, can be computed from equations (1.24) and (1.25). The growth rate g can be determined from (1.26).

1.2.2 Properties of decentralized equilibrium

We require, for the relevant balanced growth path, X^* satisfies the condition of non-negativeness

$$X^* = (1-\tau)\left(\frac{g}{T-\theta}\right)^{\frac{\alpha-1}{\alpha}} - g \geq 0. \tag{1.27}$$

It is useful to note that X^* in equation (1.27) is a continuous and decreasing function of g. The function X^* approaches to infinity as g tends to zero and thus X^* can be negative for a high enough g. Let $\bar{g}$ satisfies $X^*(\bar{g}) = 0$ then X^* is negative for $g > \bar{g}$. The non-negativeness condition for X^* is satisfied provided the growth rate g satisfies $g < \bar{g}$ and $X^*(\bar{g}) = 0$ with $\bar{g} = \frac{(1-\tau)^{\alpha}}{(T-\theta)^{\alpha-1}}$. It is interesting to mention here that as X^* is a decreasing function of g therefore the possibility of non-relevant equilibrium is associated with the high growth equilibrium.

To analyze the properties of decentralized equilibrium, we further explore (1.26). From equation (1.26), we can introduce two functions $\Gamma(g)$ and $\Psi(g)$

defined as

$$\Gamma(g) = \alpha(1-\tau)\left(\frac{g}{T-\theta}\right)^{\frac{\alpha-1}{\alpha}}, \tag{1.28}$$

$$\Psi(g) = [\sigma - (1-\sigma)\gamma]g + \rho. \tag{1.29}$$

The function $\Gamma(g)$ satisfies following properties:

$$\lim_{t\to 0}\Gamma(g) = \infty, \lim_{t\to 0}\Gamma'(g) = -\infty, \lim_{t\to\infty}\Gamma(g) = 0, \lim_{t\to\infty}\Gamma'(g) = 0, \tag{1.30}$$

and thus is a strictly decreasing and strictly convex function of the growth rate g. The function $\Psi(g)$ is a linear function of g and slope is dependent on σ. We have the following three possibilities:

$$\sigma \begin{cases} > \frac{\gamma}{1+\gamma} & \text{slope of } \Psi(g) \text{ is positive} \\ = \frac{\gamma}{1+\gamma} & \text{slope of } \Psi(g) \text{ coincides with horizontal line at } \rho \\ < \frac{\gamma}{1+\gamma} & \text{slope of } \Psi(g) \text{ is negative} \end{cases}$$

We state conditions of unique and multiple equilibria in following proposition:

Proposition : *A unique long-run equilibrium exists for $\sigma \geq \frac{\gamma}{1+\gamma}$ and the possibility that the multiple equilibria exist requires $\sigma < \frac{\gamma}{1+\gamma}$.*

Proposition states that a unique long-run equilibrium exists when function $\Psi(g)$ has positive or zero slope (see Figure 1.1). For the emergence of multiple equilibria the slope of function $\Psi(g)$ is negative and we have three possibilities for BGP (i) No solution exist (ii) single solution exist and (iii) two solutions exist. Thus for the negative slope of $\Psi(g)$ we can have no BGP, unique BGP or multiple BGPs. The emergence of multiple equilibria is possible for case (iii). For the multiple equilibria case, there is one steady state (Point E in Figure 1.2) characterized by low growth and one steady state (Point F in Figure 1.2) characterized by high growth. The Proposition explains that countries will experience low growth unique steady state equilibrium when they give a higher value to present consumption as they have placed a higher value to inverse of intertemporal elasticity of substitution while a smaller value of inverse of

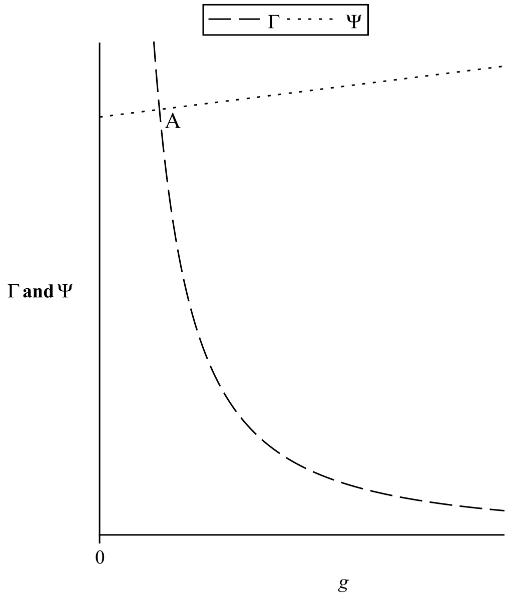


Figure 1.1: Unique equilibrium, $\sigma \geq \frac{\gamma}{1+\gamma}$

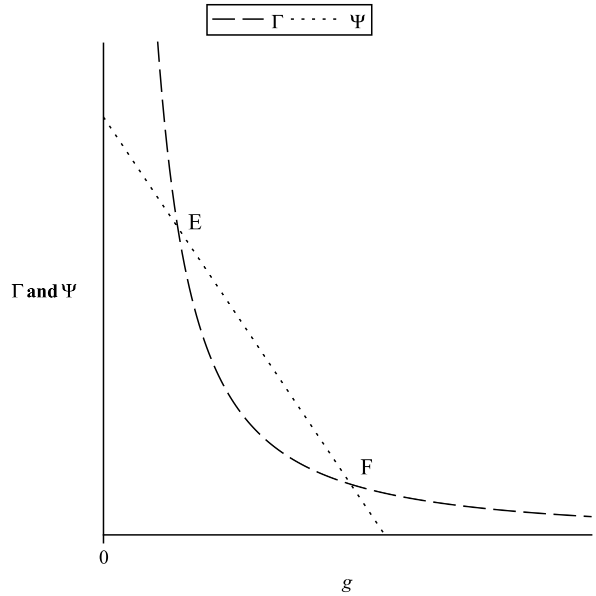


Figure 1.2: Multiple equilibria, $\sigma < \frac{\gamma}{1+\gamma}$

intertemporal elasticity of substitution will yield in multiple equilibria. So we can conclude that the number of equilibria is influenced by the preferences made by the individual agents. In order to capture the case of developing and developed world we perform simulations and change preference parameter (weight of the environmental quality) and examine its effect on growth rates. So we formulated a range of environmental quality that apprehended unique and multiple equilibria. For example $\sigma \geq \frac{\gamma}{1+\gamma}$ shows that inverse of intertemporal elasticity of substitution is higher than weight assigned to environmental quality, which means that agents are putting a higher value on current consumption and so will consume more in present and will invest less in physical capital and environmental quality that results in low growth equilibrium.

The emergence of multiple equilibria can be analyzed as follows. Private agents in a decentralized economy fail to judge the evolution of environment quality and thus they will invest more in their physical capital in order to have a high level of environment quality. This leads to higher income, higher tax revenue and as a result a higher level of environment quality is supplied to have high growth equilibrium. On the contrary if negative expectations are formed about the environmental quality the economy attains low growth equilibrium because agents will increase their consumption and invest less in physical environmental quality.

1.3 Numerical simulations

In this section, we employ numerical simulations to explore the theoretical results further and we consider following two scenarios:
Scenario 1: When capital share α is equal to the intertemporal elasticity of substitution σ i.e. $\alpha = \sigma$
Scenario 2: $\alpha \neq \sigma$

1.3.1 Numerical simulations under parameter restriction $\alpha = \sigma$

For the unique equilibrium case with $\sigma \geq \frac{\gamma}{1+\gamma}$ we use benchmark values of parameters $\alpha = 0.35,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.35,\ \gamma = 0.5$ then we obtain $\bar{g} = 0.0399,\ g = 0.0104$. It is important to mention here that if we fix all other parameters and vary the weight of environmental quality γ in utility function satisfying restriction $\sigma \geq \frac{\gamma}{1+\gamma}$, a unique equilibrium exist for $0 < \gamma \leq 0.53$.

Now we analyze the case $\sigma < \frac{\gamma}{1+\gamma}$, we use the benchmark values as are used for unique equilibrium case and we vary weight of environmental quality γ. We arrive at following three cases: (i) For $0.54 \leq \gamma \leq 6.5$, we obtain two BGPs but the high growth path fails to satisfy the condition of non-negativeness. We conclude that only one relevant BGP exists for this case (ii) For $6.6 \leq \gamma \leq 9.8$, two relevant BGPs exist (iii) For $\gamma > 9.8$ no BGP exists.

Table 1.1: Multiple equilibria: $6.6 \leq \gamma \leq 9.8$

γ	$\bar{g}$	g_l	g_h
6.6	0.0399	0.0125	0.0395
8.5	0.0399	0.0140	0.0274
9.8	0.0399	0.0174	0.0193

The weight of environmental quality in utility function is sufficiently high for the multiple equilibria case. For the multiple equilibria case with benchmark values of parameters $\alpha = 0.35,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.35$, the effect of change in parameter γ on low and high growth equilibria are demonstrated in Table 1.1 and Figure 1.3, we denote low growth equilibrium by g_l and high growth by g_h. For multiple equilibria case, $6.6 \leq \gamma \leq 9.8$, if we set $\gamma = 6.6$ then we obtain $\bar{g} = 0.05097,\ g_l = 0.0125,\ g_h = 0.0395$. To ensure the non-negativeness of X^* under a given set of parameters, we need the economic growth rate to be less or equal to 5.09% i.e. $\bar{g}$. At the steady state the two growth rates are 1.25% and 3.95% and both growth rates satisfy the condition of non-negativeness. The growth gap between two growth rates is 2.7%. If we fix $\gamma = 8.5\ g_l = 0.0140,\ g_h = 0.0274$ and gap between two growth rates reduces to 1.34%. If we increase γ to 9.8 then $g_l = 0.0174,\ g_h = 0.0193$ and the gap between two growth rate reduces to 0.19% . We conclude that an increase in γ results in increase in low growth equilibrium g_l. On the other hand, the high growth equilibrium g_h decreases with increase in weight of environmental quality. It is worthy to mention here that $6.6 \leq \gamma \leq 9.8$ the low growth equilibrium $0.0125 \leq g_l \leq 0.0174$ and the high growth equilibrium $0.0193 \leq g_h \leq 0.0395$. The growth gap between two equilibria for $\gamma = 6.6$ is 2.7% and it reduces to 0.19% if we set $\gamma = 9.8$. The gap between growth rates decreases when we take higher values of weight of environment quality in the utility function, this is extremely interesting result in developing country

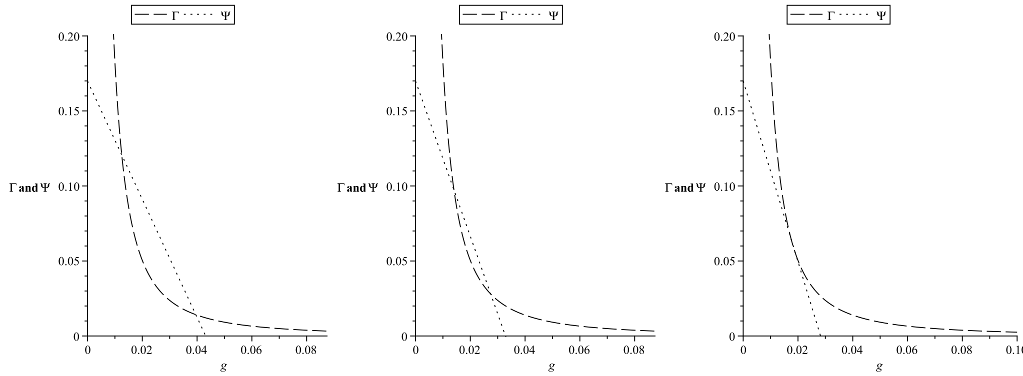


Figure 1.3: Effect of change in γ on equilibria for $\alpha = \sigma$

context.

1.3.2 Numerical computations for $\alpha \neq \sigma$

Now we relax the assumption that the capital share equals to the inverse of intertemporal elasticity of substitution and study the unique and multiple equilibria for the fairly general values of capital share and intertemporal elasticity of substitution. We use benchmark values of parameters $\alpha = 0.4,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.5,\ \gamma = 0.7$ then we obtain $\bar{g} = 0.05097,\ g = 0.0123$. It is important to mention here that if we fix all other parameters and vary the weight of environmental quality γ in utility function satisfying restriction $\sigma \geq \frac{\gamma}{1+\gamma}$, a unique equilibrium exist for $0 < \gamma \leq 1$.

Now we analyze the case $\sigma < \frac{\gamma}{1+\gamma}$, we use the benchmark values as are used for unique equilibrium case and we vary weight of environmental quality γ. We arrive at following three cases: (i) For $0.54 \leq \gamma \leq 6.8$, we obtain two BGPs but the high growth path fails to satisfy the condition of non-negativeness. We conclude that only one relevant BGP exists for this case (ii) For $6.9 \leq \gamma \leq 9.9$, two relevant BGPs exist (iii) For $\gamma > 9.9$ no BGP exists. The weight of environmental quality in utility function is sufficiently high for the multiple equilibria case. For multiple equilibria case, $6.9 \leq \gamma \leq 9.9$, for benchmark values of parameters the effect of change in parameter γ on low and high growth qeuiliberia is presented in Table 1.2 and Figure 1.4. If we set $\gamma = 6.9$ then we obtain $\bar{g} = 0.05097,\ g_l = 0.0152,\ g_h = 0.0507$. To ensure the non-negativeness of X^* under a given set of parameters, we need the economic growth rate to be less or equal to 5.097% i.e. $\bar{g}$. At the steady state the two growth rates are 1.52% and 5.07% and both growth rates satisfy the condition of non-negativeness.The growth gap between two growth rates is 3.55%. Now if we set $\gamma = 7.8$ then we obtain $\bar{g} = 0.05097,\ g_l = 0.01604,\ g_h = 0.04198$. The growth gap between two equilibria is 2.594%. If we take $\gamma = 9.9$, we obtain $g_l = 0.02163,\ g_h = 0.02419$ then growth gap between growth rates reduces to 0.256%. As low growth countries growth rate will increase meanwhile high growth countries will experience decrease in growth. This will be due to decline in marginal utility or it's because marginal utility of environmental quality is negative beyond a certain point. So increase in weight of the environmental quality will decrease the growth of developed world and after this point there will be no growth equilibrium. Now we analyze the case $\sigma < \frac{\gamma}{1+\gamma}$, we use the benchmark values as are used for unique equilibrium case and we vary weight of environmental quality γ. We arrive at following three cases: (i) For $0.54 \leq \gamma \leq 6.8$, we obtain two BGPs but the high growth path fails to satisfy the condition of non-negativeness. We conclude that only one relevant BGP exists for this case (ii) For $6.9 \leq \gamma \leq 9.9$, two relevant BGPs exist (iii) For $\gamma > 9.9$ no BGP exists. The weight of environmental quality in utility function is sufficiently high for the multiple equilibria case. For the multiple equilibria case with benchmark values of parameters $\alpha = 0.4,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.5$, the effect of change in parameter γ on low and high growth equilibria is demonstrated in Table 1.2, we denote low growth equilibrium by g_l and high growth by g_h. For this set of parameters, our model yields multiple equilibria if the weight of environmental quality in utility function is chooser from $6.9 \leq \gamma \leq 9.9$. A closer look at Table 1.2

Table 1.2: Multiple equilibria: $6.9 \leq \gamma \leq 9.9$

$\alpha = 0.4,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.5$

γ	$\bar{g}$	g_l	g_h
6.9	0.0399	0.0152	0.0507
7.8		0.01604	0.04198
9.9		0.02163	0.02419

shows that an increase in γ results in increase in low growth equilibrium g_l. On the other hand, the high growth equilibrium g_h declines with increase in weight of environmental quality. To ensure the non-negativeness of X^* under a given set of parameters, we need the economic growth rate to be less or equal to 5.097% i.e. $\bar{g}$. At the steady state the two growth rates are 1.52% and 5.07% and both growth rates satisfy the condition of non-negativeness.The growth gap between two growth rates is 3.55%. Now if we set $\gamma = 7.8$ then we obtain $\bar{g} = 0.05097,\ g_l = 0.01604,\ g_h = 0.04198$. The growth gap between two equilibria is 2.594%. If we take $\gamma = 9.9$, we obtain $g_l = 0.02163,\ g_h = 0.02419$ then growth gap between growth rates reduces to 0.256%.

The gap between growth rates decreases when we increase the weight of environment quality in utility function. This holds true for fairly general values of capital share and intertemporal elasticity of substitution.

How can we determine that the multiple equilibria derived here are stable and meaningful? If only one steady state in case of multiple equilibria is meaningful then how we can utilize it to improve the low-growth equilibrium? To answer these questions we perform stability analysis in next chapter. The main purpose of the stability analysis is to determine that whether the multiple equilibria derived here are stable and meaningful. This will help the policy makers to analyze the requirements for environment quality in feasibility and policy validity perspective.

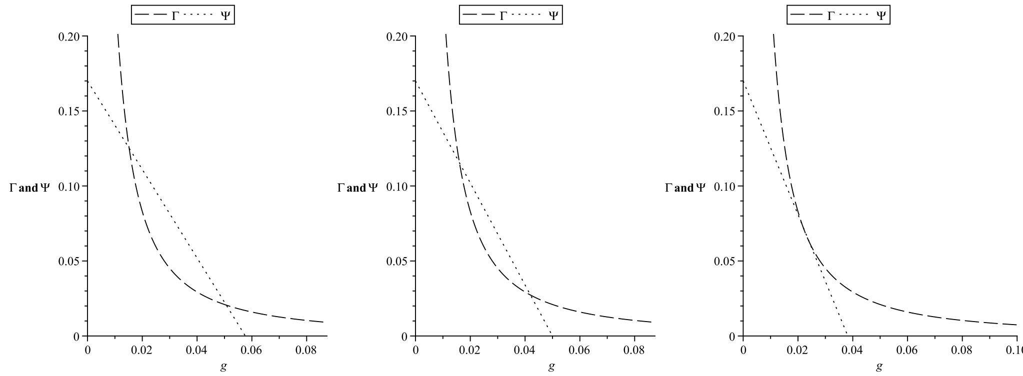


Figure 1.4: Effect of change in γ on equilibria for $\alpha \neq \sigma$

In this chapter, we have explained in detail framework of the model and explained how we derive our equation of motions and then solved our dynamic model for growth rate of consumption, environmental quality and physical capital. We found that the growth rate of consumption is directly proportional to the growth rate of environmental quality and how environmental quality is dependent upon abatement expenditures. After solving the model we have found certain restrictions in which unique and multiple equilibrium exists. Such as $\sigma > \frac{\gamma}{1+\gamma}$ yields unique equilibrium and $\sigma < \frac{\gamma}{1+\gamma}$ will result in multiple equilibrium. By changing the weight assigned to environmental quality we have found different growth regimes which explains that preferences or control variable play a crucial role in obtaining multiple growths. And we can achieve high growth by prioritizing environmental quality.

Chapter 2

Stability Analysis and Transitional Dynamics in Macroeconomic Model with Environment

In this chapter, we analyze the stability and transitional dynamics around equilibrium. We have calculated a range for the weight assigned to the environmental quality, and by changing this preference parameter we have observed stable and unstable growth paths. We have discussed two cases through numerical simulations. In the first case we have assumed that one structural parameter (inverse of intertemporal elasticity of substitution(σ)) is equal to one preference parameter (weight assigned to environmental quality (γ). And in the second case we have removed this restriction and solved for stability.

2.1 Stability and transitional dynamics around equilibrium

In order to examine the local stability of the BGP, we linearize the reduced dynamic system of equations (15) and (16), associated with the original system of equations (12) − (14), around steady state values denoting X^* and Z^* (refer to Appendix D). The Jacobian matrix of linearization can be derived from

$$J\mid_{(X^*,Z^*)}=\begin{bmatrix} \frac{\partial \dot{X}}{\partial X}\mid_{(X^*,Z^*)} & \frac{\partial \dot{X}}{\partial Z}\mid_{(X^*,Z^*)} \\ \frac{\partial \dot{Z}}{\partial X}\mid_{(X^*,Z^*)} & \frac{\partial \dot{Z}}{\partial Z}\mid_{(X^*,Z^*)} \end{bmatrix}, \tag{2.1}$$

and this results in

$$J^* = \begin{bmatrix} X^* & a_{12} \\ -Z^* & (\alpha-1)X^* - (T-\theta)Z^{*\alpha} \end{bmatrix}, \tag{2.2}$$

where a_{12} is given by

$$a_{12} = \frac{1}{\sigma}X^*(Z^*)^{\alpha-2}[\gamma(1-\sigma)(T-\theta)\alpha Z^* + (\alpha-1)(1-\tau)(\alpha-\sigma)]. \tag{2.3}$$

Finally, we have following linear system:

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} X^* & a_{12} \\ -Z^* & (\alpha-1)X^* - (T-\theta)Z^{*\alpha} \end{bmatrix} \begin{bmatrix} X - X^* \\ Z - Z^* \end{bmatrix}. \tag{2.4}$$

Next, we discuss the local dynamics of the system which includes one control (jump) variable X and one state variable Z and discuss stability of this system (See Appendix E) We obtain following two eigenvalues of Jacobian matrix J^*:

$$\lambda^1 = \frac{1}{2}[tr(J^*) + \sqrt{tr(J^*)^2 - 4det(J^*)}], \tag{2.5}$$

$$\lambda^2 = \frac{1}{2}[tr(J^*) - \sqrt{tr(J^*)^2 - 4det(J^*)}], \tag{2.6}$$

where

$$tr(J^*) = \alpha X^* - (T-\theta)(Z^*)^\alpha, \tag{2.7}$$

and the determinant of Jacobian matrix is

$$detJ^* = (\alpha - 1)(X^*)^2 - (T - \theta)(Z^*)^\alpha X^* +$$
$$\frac{1}{\sigma} X^*(Z^*)^{\alpha-1}[\gamma(1-\sigma)(T-\theta)\alpha Z^* + (\alpha - 1)(1-\tau)(\alpha - \sigma)]. \tag{2.8}$$

It is difficult to determine the signs of $tr(J^*)$, $det(J^*)$ and eigenvalues but numerical simulations can be used to observe the local dynamic properties.

2.2 Numerical simulations

In this section, we consider numerical simulations for the ratio of consumption to capital stock X, the ratio of capital stock to environmental quality Z, the ratio of consumption to environmental quality W, the growth rate g, trace and determinant of Jacobian matrix J^*. We use numerical simulations to explore the results for following two scenarios:
Scenario 1: When capital share α is equal to the intertemporal elasticity of substitution σ i.e. $\alpha = \sigma$
Scenario 2: $\alpha \neq \sigma$

2.2.1 Numerical simulations for Scenario 1

We fix parameter values at $\alpha = 0.35,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.35$ and choose different values of the weight of environmental quality γ in the utility function. The results of numerical simulations for the case when $\alpha = \sigma$

are presented in Table 2.1 and graphically in Figures 2.1-2.4. We obtain a low growth equilibrium when environmental quality γ is given a lower weight in the representative's utility function. The equilibrium values of variables $(X = \frac{C}{K})$, $(Z = \frac{K}{E})$ and $(W = \frac{C}{E})$, the growth rate g, trace and determinants

Table 2.1: Unique and Multiple equilibria

$\alpha = 0.35,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.35$

γ	Steady states	g	$\lvert J^*\rvert$ & T^*	Eigenvalues
0.5	$X^* = 0.476073$	$g = 0.010382$	$\lvert J^*\rvert = -0.150656$	$\lambda^1 = 0.4740$
	$Z^* = 2.845748$		$T^* = 0.156243$	$\lambda^2 = -0.3178$
	$W^* = 1.354784$			
6.6	$X_l^* = 0.3326$	$g_l = 0.0125$	$\lvert J^*\rvert_l = -0.0582$	$\lambda_l^1 = 0.2988$
	$Z_l^* = 4.8258$		$T_l^* = 0.1039$	$\lambda_l^2 = -0.1949$
	$W_l^* = 1.6051$			
	$X_h^* = 0.0011$	$g_h = 0.0395$	$\lvert J^*\rvert_h = 0.0001$	$\lambda_h^1 = -0.0038$
	$Z_h^* = 129.8556$		$T_h^* = -0.0392$	$\lambda_h^2 = -0.0353$
	$W_h^* = 0.1375$			
7	$X_l^* = 0.3202$	$g_l = 0.01274$	$\lvert J^*\rvert_l = -0.0522$	$\lambda_l^1 = 0.2834$
	$Z_l^* = 5.1009$		$T_l^* = 0.0993$	$\lambda_l^2 = -0.1840$
	$W_l^* = 1.6331$			
	$X_h^* = 0.0104$	$g_h = 0.0366$	$\lvert J^*\rvert_h = 0.0013$	$\lambda_h^1 = -0.0165 + 0.0317\iota$
	$Z_h^* = 103.8315$		$T_h^* = -0.0329$	$\lambda_h^2 = -0.0165 - 0.0317\iota$
	$W_h^* = 1.0787$			
9.1	$X_l^* = 0.2340$	$g_l = 0.0149$	$\lvert J^*\rvert_l = -0.0185$	$\lambda_l^1 = 0.1735$
	$Z_l^* = 7.9780$		$T_l^* = 0.0670$	$\lambda_l^2 = -0.1065$
	$W_l^* = 1.8670$			
	$X_h^* = 0.07697$	$g_h = 0.0242$	$\lvert J^*\rvert_h = -.0053$	$\lambda_h^1 = 0.0014 + 0.0728\iota$
	$Z_h^* = 31.8813$		$T_h^* = 0.0028$	$\lambda_h^2 = 0.0014 - 0.0728\iota$
	$W_h^* = 2.4537$			

of Jacobian matrix are given in first part of Table 2.1. This unique low growth equilibrium is saddle path stable as $|J^*| < 0$ and $T^* > 0$. For unique equilibrium case functions Γ and Ψ intersect at one steady state value (see Figure 2.1).

The multiple equilibria exist for the case when the weight of the environmental quality is increased. The equilibrium values of variables ($X = \frac{C}{K}$), ($Z = \frac{K}{E}$), $W = \frac{C}{E}$, the growth rate g, trace and determinants of Jacobian matrix are given in the second part of Table 2.1. We use subscript h and l to denote all variables for the high and low growth cases respectively. The low growth equilibrium is saddle path stable as $|J^*| < 0$ and $T^* > 0$. For the high growth case $|J^*| > 0$ and $T^* < 0$ which implies the high growth equilibrium is stable. The numerical values show that $X_l^* > X_h^*$, the consumption as a proportion of capital stock (X^*) is higher in the low growth case as compared to the high growth case. Similarly, the consumption as a proportion of environmental quality (W^*) is higher in the low growth case as compared to the high growth case i.e. $W_l^* > W_h^*$, and ratio of physical capital to environmental quality is much higher in high growth case then low growth case $Z_h^* > Z_l^*$ because environmental quality is used as an input in the production function. In the multiple equilibrium case we have two growth cases although environmental quality is given a higher weight still one country is at a lower growth and the other attains the higher growth is because private agents cannot know the environmental quality in an economy so if agents expect a better environmental quality they will invest more in physical capital and it will eventually lead to higher income and taxes. Higher taxes can be used for abatement expenditures which will further improve the environmental quality and will result in high growth equilibrium. In case of low growth equilibrium, agents have formed adverse expectations about the future environmental quality so they are consuming more and investing less in capital and we can see in Table 2.1 we have a higher consumption to environmental quality for low growth case

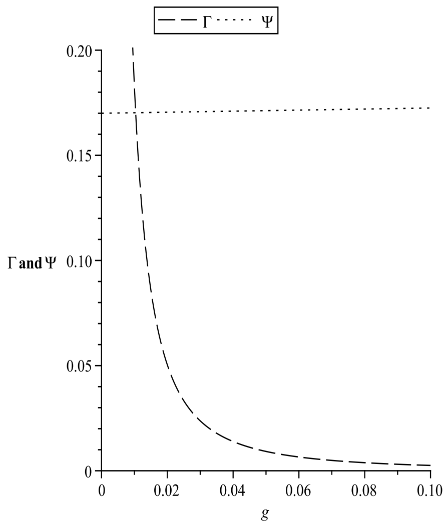


Figure 2.1: Unique equilibria,$\gamma = 0.5$

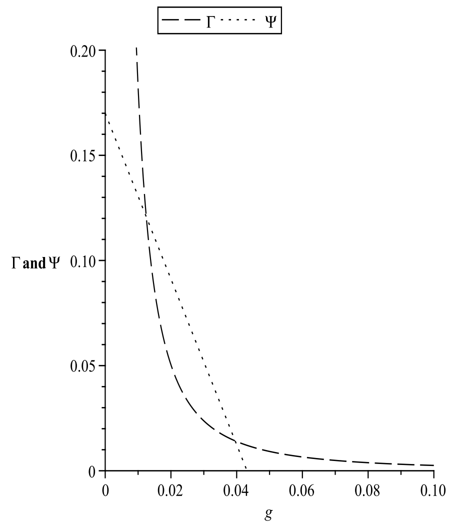


Figure 2.2: Multiple equilibria,$\gamma = 6.6$

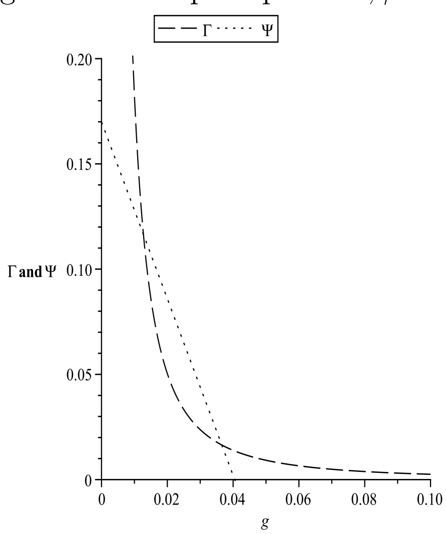


Figure 2.3: Multiple equilibria,$\gamma = 7$

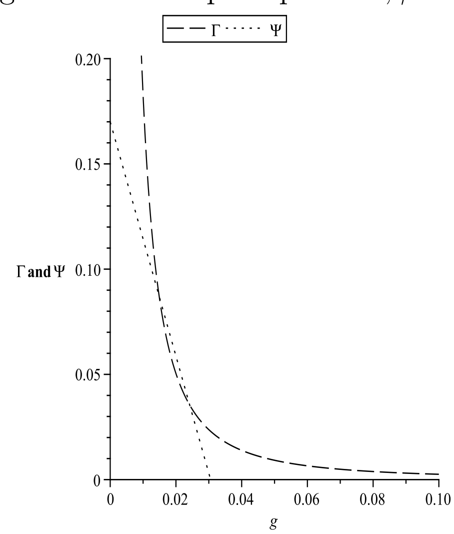


Figure 2.4: Multiple equilibria,$\gamma = 9.1$

as well thus caught in a poverty trap. For multiple equilibria case functions Γ and Ψ intersect at two steady state values (see Figures 2.2-2.4).

2.2.2 Numerical simulations for Scenario 2: $\alpha \neq \sigma$

We consider the numerical simulations for the case when $\alpha \neq \sigma$. We fix parameter values at $\alpha = 0.4$, $T - \theta = 0.0072$, $\rho = 0.17$, $\tau = 0.04$, $\sigma = 0.5$ and choose different values of the weight of environmental quality γ in the utility function. The results of numerical simulations for the case when $\alpha \neq \sigma$

are presented in Table 2.2 and graphically demonstrated in Figures 2.4-2.8. We obtain a low growth equilibrium when environmental quality γ is given a lower weight in the representative's utility function and multiple equilibria arise when environmental quality γ is given a higher weight in the representative's utility function. The unique low growth equilibrium is saddle path stable as $|J^*| < 0$ and $T^* > 0$. For the multiple equilibria case the low growth equilibrium is shown to be the saddle path stable whereas the high growth equilibrium is shown to be the stable. The numerical values witness that $X_l^* > X_h^*$ and $W_l^* > W_h^*$ for scenario 2 as well.

For both scenarios, the results of model are extremely interesting and will be useful to explain economic growth of different countries. For example if environmental quality is given relatively less importance by the agents so the agents will invest less in environmental quality (which is also an input in the production) and consume more then they will get caught in low growth, high consumption poverty traps which is the case for developing countries. The economies can potentially reach a relatively low consumption, high growth steady state if they give greater importance to environmental quality. These results are true for both $\alpha = \sigma$ and $\alpha \neq \sigma$ cases.

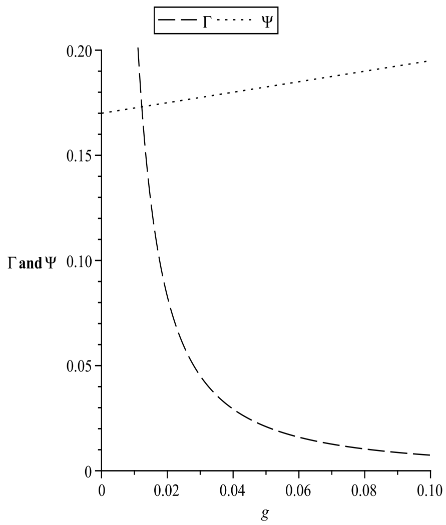


Figure 2.5: Unique equilibria,$\gamma = 0.5$

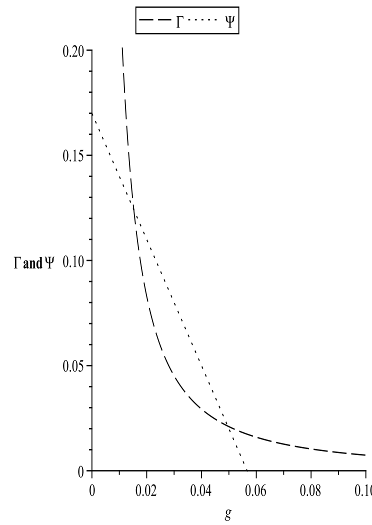


Figure 2.6: Multiple equilibria,$\gamma = 7$

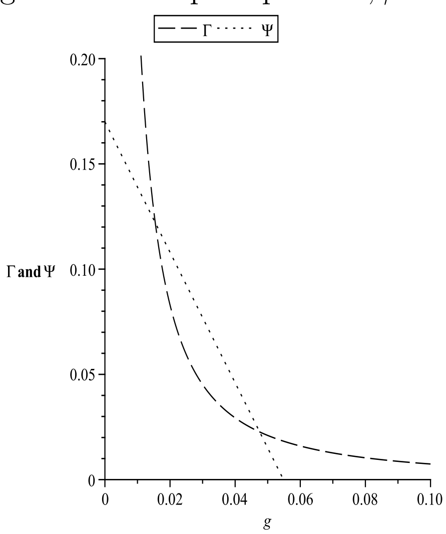


Figure 2.7: Multiple equilibria,$\gamma = 7.2$

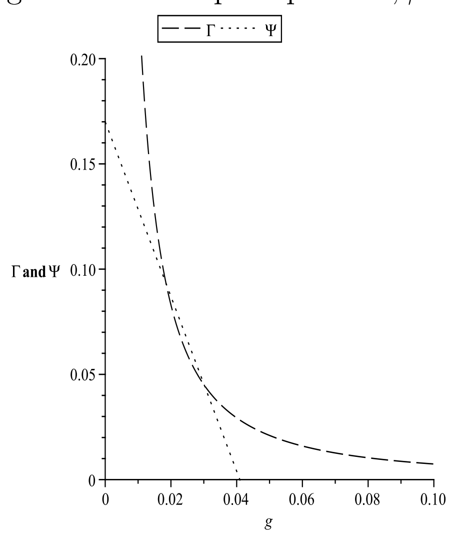


Figure 2.8: Multiple equilibria,$\gamma = 9.3$

Table 2.2: Multiple and unique equilibria $\alpha \neq \sigma$,

$\alpha = 0.4,\ T - \theta = 0.0072,\ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.5$

γ	Steady states	g	$\|J^*\|$ & T^*	Eigenvalues
0.5	$X^* = 0.420406$	$g = 0.012248$	$\|J^*\| = -0.088337$	$\lambda^1 = 0.3852$
	$Z^* = 3.774695$		$T^* = 0.155914$	$\lambda^2 = -0.2293$
	$W^* = 1.58690$			
7	$X_l^* = 0.2951$	$g_l = 0.01529$	$\|J^*\|_l = -0.0331$	$\lambda_l^1 = 0.2405$
	$Z_l^* = 6.5666$		$T_l^* = 0.1027$	$\lambda_l^2 = -0.1378$
	$W_l^* = 1.9376$			
	$X_h^* = 0.0035$	$g_h = 0.0496$	$\|J^*\|_h = 0.00033$	$\lambda_h^1 = -0.0083$
	$Z_h^* = 124.4580$		$T_h^* = -0.0482$	$\lambda_h^2 = -0.0399$
	$W_h^* = 0.4401$			
7.2	$X_l^* = 0.2898$	$g_l = 0.0155$	$\|J^*\|_l = -0.0313$	$\lambda_l^1 = 0.2342$
	$Z_l^* = 6.7523$		$T_l^* = 0.1004$	$\lambda_l^2 = -0.1338$
	$W_l^* = 1.9565$			
	$X_h^* = 0.0091$	$g_h = 0.0475$	$\|J^*\|_h = 0.0008$	$\lambda_h^1 = -0.0220 + 0.0184\iota$
	$Z_h^* = 112.0083$		$T_h^* = -0.0439$	$\lambda_h^2 = -0.0220 - 0.0184\iota$
	$W_h^* = 1.0137$			
9.3	$X_l^* = 0.2145$	$g_l = 0.0185$	$\|J^*\|_l = -0.0108$	$\lambda_l^1 = 0.14298$
	$Z_l^* = 10.5863$		$T_l^* = 0.0673$	$\lambda_l^2 = -0.0757$
	$W_l^* = 2.2711$			
	$X_h^* = 0.0817$	$g_h = 0.0302$	$\|J^*\|_h = 0.0038$	$\lambda_h^1 = 0.0012 + 0.0616\iota$
	$Z_h^* = 35.9835$		$T_h^* = 0.0024$	$\lambda_h^2 = 0.0012 - 0.0616\iota$
	$W_h^* = 2.9384$			

2.3 Numerical Simulation: Weight of environmental quality

For different ranges of weight of environmental quality(γ) we have found very interesting results. If $0 \leq \gamma \leq 0.53$ we have found unique low growth equilibrium in which we have obtain a higher value of X which means that countries who will fall in this range have consumption and are investing less in physical capital and environmental quality which is a case of developing countries. For values $0.54 \leq \gamma \leq 6.5$ we have obtain multiple equilibria but is not saddle path stable so countries who will fall in this range will not have a stable growth and will also experience negative values for X and E which means negative consumption.

If the weight of environmental quality lies between the range of $6.6 \leq \gamma \leq 7.3$ we have obtain multiple growth equilibria for the same value of γ a higher and lower growth cases can be witnessed. In multiple equilibria case we have $X_l > X_h$ and $W_l > W_h$ which shows that in lower growth equilibrium a country will get caught in high consumption poverty trap because agents will invest less in environmental quality and consume more because agents can not know the true environmental quality in the economy.

2.3.1 Cyclical Dynamics

If the weight of environmental quality ranges between $7.4 \leq \gamma \leq 9.8$ then we will experience multiple equilibria but in this case high g_h and low growth g_l will start to converge because if too much weight is given to environmental quality then it will also affect growth of country. So in this scenario $X_l > X_h$ but $W_l < W_h$ so for this particular case we can say that cyclical dynamics exist and two countries growth rates are starting to converge if environmental quality is given too much weight and it is a real life implication. When environmental quality is given more higher weight $\gamma > 9.8$ then we will not experience any growth regime or we can say that no growth will exist.

2.4 Multiple Equilibria and Policy Implications

As discussed in the prior section, our model yields unique and multiple equilibria. We have also established that agent's preferences play a crucial role in determining these growth regimes. When agents prioritized environmental quality we observed convergence between low and high growth equilibria. But in developing countries' context recognizing the importance of environmental quality can take considerable amount of time. So in that case government can play a fundamental role in improving environmental quality by increasing abatement expenditures or reducing emissions and can impact long run growth.

2.4.1 Policy Implications in cross-country comparison when $\alpha = \sigma$

By considering similar values of the parameter as above and performing numerical simulations for case $\alpha = \sigma$ to analyze the changes in the difference between abatement expenditure and emissions $(T - \theta)$, we got the results shown in Tables 2.3 and 2.4. In Table 2.3 we can compare two countries. In the first case weight given to environmental quality is high while in the second case, we have higher value for the difference between abatement and emissions. We can see that in latter one we have higher values for both high and low growth equilibria as compared to the first country. This result is important because it indicates a high growth even when agents are not giving higher weight to the environmental quality. So we can infer that agent's self-fulfilling expectation (adverse) regarding future environmental quality can be attenuated by higher abatement expenditures. Because in this case government will take the lead and will make rules and regulations for abatement expenditures and emissions to ensure better environmental quality. Figure 2.9 is a graphical illustration of Table 2.3. We can observe the multiple equilibria in both cases as the intersection takes place at two points. We can distinguish from Figure 2.9 a and 2.9 b that the intersection line slope is steeper when environmental quality is high and abatement expenditure is low but in a reverse case in which we have incident of higher abatement expenditures and lower weight given to environmental quality the slope is flatter. Which also indicate that the difference between high and low growth equilibrium in latter is more than the first.

2.4.2 Policy implications in improving the inferior equilibrium for $\alpha = \sigma$

In Table 2.4 we have performed similar simulations but in this case both countries have been assigned a higher weight to the environmental quality. We can make comparison between Table 2.3 and 2.4 , as Table 2.3 illustrates developing countries case while Table 2.4 shows developed countries in which agents give more emphasis to the environmental quality which causes the gap between low and high growth to shrink. When an economy develops, individuals will put more weight to the environmental quality which improves low growth equilibrium and shrinks the gap between low and high growth, thus stabilizes economic growth. Figure 2.10 is a graphical illustration of Table 2.4. From these graphs we can analyze that slope of straight line is steeper than the figure 2.9. The steeper line means slope is higher which shows that difference between high and low growth will be smaller. It's due to the convergence of

Table 2.3: Effects of Change in $T - \theta$

$\alpha = 0.35,\ \ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.35$

I	γ	$T-\theta$	g_l	g_h
$6.6 \leq \gamma \leq 9.8$	6.6	0.0072	0.0125	0.0395
$5.3 \leq \gamma \leq 7.2$	5.3	0.01	0.0178	0.0493

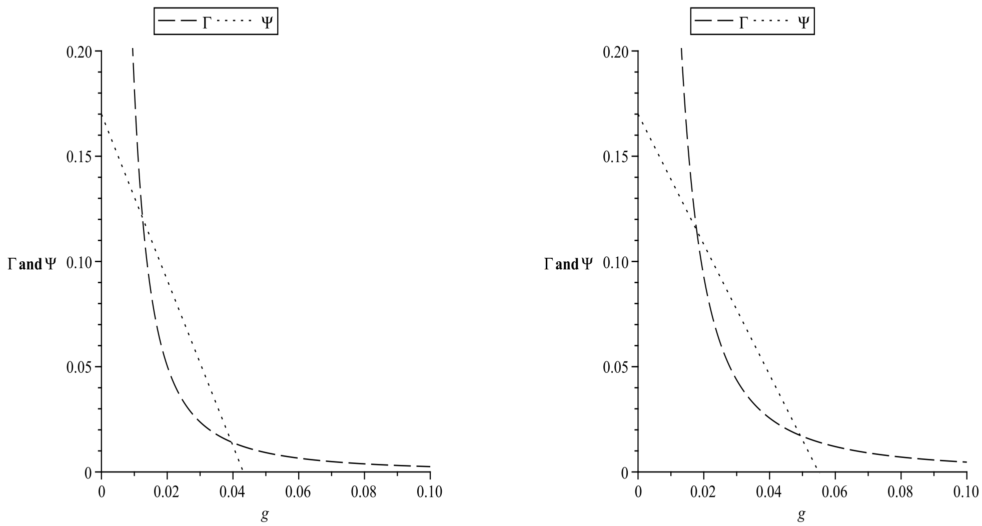


Figure 2.9: Effect of change in γ, T and θ in cross country comparison for $\alpha = \sigma$

high and low growth equilibrium when countries will assign higher weight to the environmental quality. In this case low growth equilibrium is rising while high growth equilibrium is decreasing. We can conclude that increasing the weight of environmental quality over a time span will result in improving the low growth equilibrium or inferior equilibrium.

Table 2.4: Effects of Change in $T - \theta$

$\alpha = 0.35, \;\; \rho = 0.17, \; \tau = 0.04, \; \sigma = 0.35$

I	γ	$T - \theta$	g_l	g_h
$6.6 \leq \gamma \leq 9.8$	9.8	0.0072	0.0174	0.0193
$5.3 \leq \gamma \leq 7.2$	7.2	0.01	0.0240	0.0270

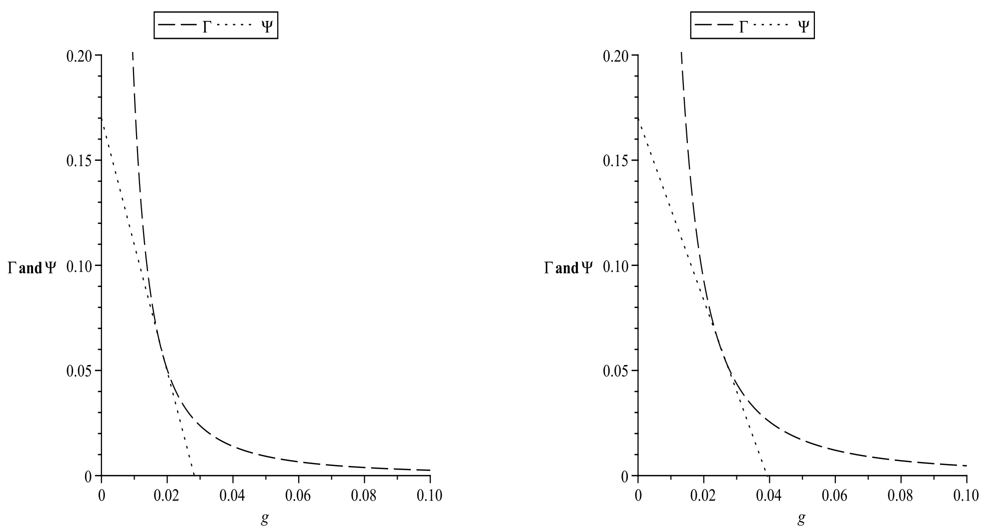


Figure 2.10: Effect of change in γ, T and θ in improving the inferior equilibrium for $\alpha = \sigma$

Table 2.5: Effects of Change in $T - \theta$

$\alpha = 0.4, \ \ \rho = 0.17, \ \tau = 0.04, \ \sigma = 0.5$

I	γ	$T - \theta$	g_l	g_h
$6.9 \leq \gamma \leq 9.9$	6.9	0.0072	0.0152	0.0507
$5.7 \leq \gamma \leq 7.4$	5.7	0.01	0.0219	0.0617

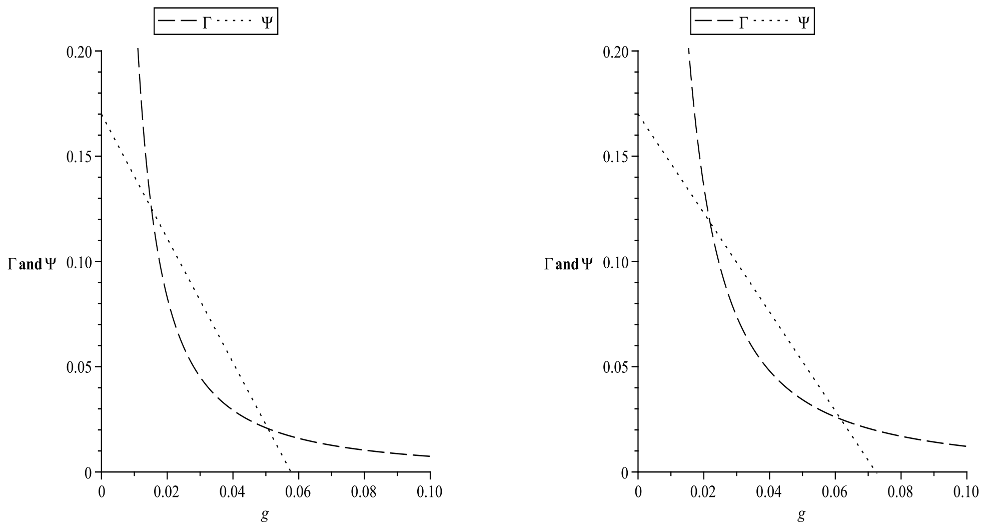


Figure 2.11: Effect of change in γ, T and θ in cross country comparison for $\alpha \neq \sigma$

2.4.3 Policy Implications in cross-country comparison when $\alpha \neq \sigma$

The numerical benchmark parameters are similar but in this case weight assigned to the intertemporal elasticity of Substitution is higher as compared to the Tables 2. 3 and 2.4 respectively. In Table 2.5, illustrate the example of two countries, in the first country higher weight is being assigned to environmental quality and in the second country case difference between abatement and emissions is higher. The latter is more relevant in context to the developing countries because the environmental quality is unrecognized so government can make laws to control level of emissions or can increase abatement expenditure. This way developing countries can achieve higher growth and government will not be hard pressed to control individual's expectations about the future environmental quality. Figure 2.11 describes Table 2.5, when inverse of intertemporal elasticity of Substitution is higher than weight assigned to physical capital. Since it's the case of a cross country comparison in which both countries assign lower weights to environmental quality but one country's abatement expenditures are higher than the other. We can analyzed in graphs two growth regimes for the same weight of environmental quality. We can see that when abatement expenditures are higher (Figure 2.11b) we have a higher growth for both high and low growth equilibrium as compare to the case when abatement expenditures are less.

2.4.4 Policy implications in improving the inferior equilibrium for $\alpha \neq \sigma$

In Table 2.6 the environmental quality is ranked higher by both countries and in this case the gap between low and high growth equilibria starts to shrink. Figure 2.12 is the graphical representation of Table 2.6. We can illustrate the significance of environmental quality as the growth gap between high and low equilibria is narrow. Because in case of multiple equilibria the most desirable outcome is the one in which both high and low growth equilibria exhibit a narrow gap or a macroeconomic policy that equalizes the two growth regimes (Hosoya, 2012). In this case low growth equilibrium seems to of benefit so we can deduce that in order to stabilize economic fluctuations there is a need to realize the significance of environmental quality by the agents in the economy. But in case of developing countries a policy is required regarding the improvement in the environmental quality because realizing the importance

Table 2.6: Effects of Change in $T - \theta$

$\alpha = 0.4,\ \ \rho = 0.17,\ \tau = 0.04,\ \sigma = 0.5$

I	γ	$T - \theta$	g_l	g_h
$6.9 \leq \gamma \leq 9.9$	9.9	0.0072	0.0206	0.0257
$5.7 \leq \gamma \leq 7.4$	7.4	0.01	0.0298	0.0339

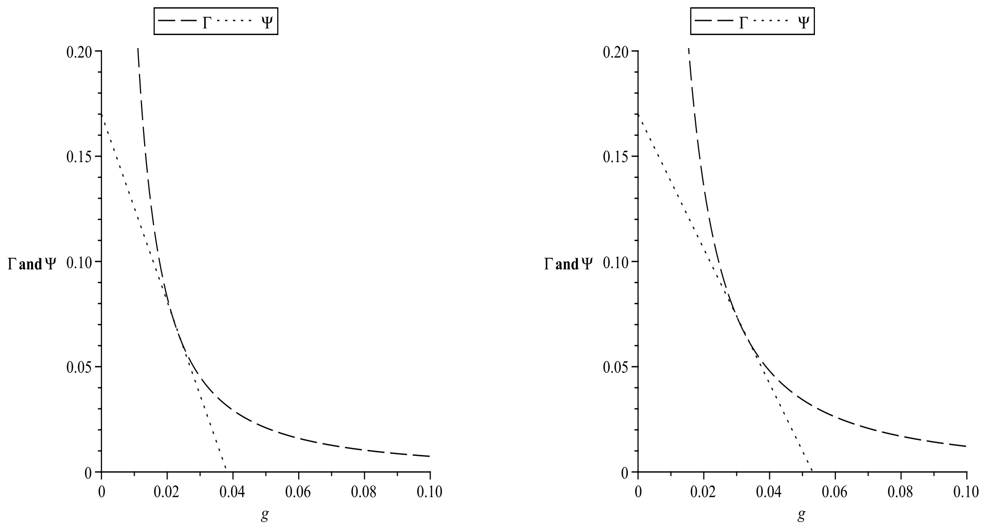


Figure 2.12: Effect of change in γ, T and θ in improving the inferior equilibrium for $\alpha \neq \sigma$

of environmental quality will be a time taking procedure and can take years. But for developing countries to come out of the poverty trap or low growth equilibrium government can ensure that the difference between abatement and emissions is higher.

Conclusions

In this paper, we developed an economic growth model which incorporates environmental quality in both the utility and the production functions. First, we assumed that the share of capital is equal to the inverse of intertemporal elasticity of substitution and after solving our model for the balanced growth path, we found cases of unique and multiple equilibria. If environmental quality is given relatively less weight in the utility function then only one steady state exists which yields a low growth rate. On the other hand, multiple equilibria exist if environmental quality is given greater weight in the utility function. These results hold true even for fairly general values of capital share and intertemporal elasticity of substitution. The results of our model show the importance of environmental quality in the process of economic growth. In particular, we have show that an economy's productive capacity is dependent on its environmental quality which means that higher environmental quality can lead to higher output while lower environmental quality can limit output in an economy. Using our framework, we showed that if individuals consider the environment to be less significant, an economy can be caught in a low growth, high consumption growth trap in which environmental quality remains low over time. On the other hand, if individuals in an economy give greater importance to environmental quality then it is possible for them to reach a low consumption, high growth and better environmental quality. So in the long run economic growth and improved environmental quality will have to go hand in hand.

In such an economy, agents do not value environmental quality and give it a lower weight due to adverse expectations about the future environmental quality and are investing less in physical capital as well by focusing more on consumption of goods. In such a case a country gets caught in high consumption lower growth equilibrium and we have witnessed a higher value for the ratio of consumption to physical capital ($X = \frac{C}{K}$) and lower value for the physical capital to environmental quality as well but ($Z = \frac{K}{E}$) because of the lower preference for environmental quality this will result in lower economic growth. But there can also be incidence of high growth equilibrium in which agents who value environmental quality and put a higher weight on the environment have experienced low ratio of consumption to physical capital ($X = \frac{C}{K}$) but a high ratio of physical capital to environment ($Z = \frac{K}{E}$) which will lead to higher income and higher abatement efforts as well. Similarly Dinda (2005) has also built an endogenous growth model in which he concludes that environmental quality has a positive effect on environmental growth which leads to an increase in marginal utility of consumption but he does not discuss how agents' preference towards environmental quality can have an impact on growth and result in low growth equilibrium. Moreover, we have also proposed policy implications for the developing countries because agents in developing world do not prioritize environmental quality, so developing countries can increase growth by making high expenditures on abatement activities or by putting restrictions on level of emissions. Increasing the gap between abatement and emission can help achieve higher growth.

Bibliography

[1] Agénor, R. P., 2008. Health and infrastructure in a model of endogenous growth. Journal of Macroeconomics 30, 1407-1422.

[2] Agénor, P. R., 2010. A theory of infrastructure-led development. Journal of Economic Dynamics and Control 34, 932-950. Alonso-Carrera, J.,

[3] Antoci, A., Galeotti, M. and Russu, P., 2014. Global analysis and indeterminacy in a sector growth model with human capital. International Journal of Economic Theory 10, 313-338

[4] Aznar- Márquez, J., and Ruiz-Tamarit, R. J., 2005. Renewable Natural Resources and Endogenous Growth. Macroeconomic Dynamics 9, 170-197.

[5] Beckerman, W., 1992. Economic Growth and the Environment: Whose Growth? Whose Environment?. University of Oxford 4, 481-496.

[6] Benhabib, J. and Perli, R., 1993. Uniqueness and Indeterminacy: Transitional Dynamics in a Model of Endogenous Growth. Journal of Economic Theory 63, 113-142.

[7] Boucekkine, R., Martinez, B., Ruiz-Tamarit, R. J., (2014). Optima Sustainable Policies Under Pollution Ceiling: the Demographic Side. Mathematical Modelling of Natural Phenomena 9, 38-64.

[8] Boucekkine, R., Ruiz-Tamarit, J. R., 2008. Special functions for the study

of economic dynamics: the case of the LucasUzawa model. Journal of Mathematical Economics 44, 33-54.

[9] Brock, A. W., Taylor, S. M., 2005. Economic Growth and Environment: A Review of Theory and Empirics. Handbook of Economic Growth, in: Philippe Aghion and Steven Durlauf (ed.), Handbook of Economic Growth 1, chapter 28, 1749-1821.

[10] Byrne, M. M., (1997). Is Growth a Dirty Word? Pollution, Abatement and Endogenous Growth. Journal of Development Economics 54, 261-284.

[11] Chen, B.L., 2007. Multiple bgps in a growth model with habit persistence. Journal of Money, Credit and Banking 39, 25.48.

[12] Chilarescu, C., 2011. On the existence and uniqueness of solution to the LucasUzawa model. Economic Modelling 28, 109-117.

[13] Common, S. M., and Barbier, B. E., 1996. Economic Growth and Environmental Degradation: The Environmental Kuznets Curve and Sustainable Development. Journal of World Development 24, 1151-1160.

[14] Dinda, S., 2005. A Theoratical Basis for the Environmental Kuznetz Curve. Journal of Ecological Economics 53, 403-413.

[15] Economides, G., Philippopoulos, A., 2008. Growth Enhancing Policy is the Means to Sustain Environment. Review of Economic Dynamics 11, 207-219.

[16] Ehrlich, P. R., and Holdren, P. J., 1971. Global Ecology: Readings Toward a Rational Strategy for Man. New York, NY:Harcourt Brace Jovanovich.

[17] Forster, B. A., 1973. Optimal Capital Accumulation in a Polluted Environment. Review of Economic Studies 39, 544-547.

[18] Freire-Serén, M., 2004. Multiple equilibria, fiscal policy, and human capital accumulation. Journal of Economic Dynamic and Control 8, 841856.

[19] Gasper, J., Vasconcelos, B. P., Afonso, O., 2014. Economic growth and multiple equlibria: A critical note. Economic Modelling 36, 157-160.

[20] Greiner, A., 2005. Fiscal Policy in an Endogenous Growth Model with Public Capital and Pollution. Japenese Economic Review 56, 67-84.

[21] Gradus, R., and Smulders, S.,(1993). The Trade-off between Environmental Care and Long-term Growth: Pollution in three Proto type Growth Models. Journal of Economics 58, 25-51

[22] Grossman, G., and Kreuger, A. B.,1995. Economic Growth and Environment. Quarterly Journal of Economics 110, 353-377.

[23] Gupta, R. M., and Barman, R. T.,2010. Health, Infratsructure, environment and Endogenous Growth. Journal of Macroeconomics 32, 657-673.

[24] Hosoya, K., 2012. Growth and multiple equilibria: A unique local dynamics. Economic Modelling 29, 1662-1665.

[25] Hosoya, K., (2012). Public health infrastructure and growth: Ways to improve the inferior equilibrium under multiple equilibria. Research in Economics 2, 194-207.

[26] Kuznets, S., 1965. Economic Growth and Structural Chang. New York, Norton.

[27] Kuznets, S., 1966. Modern Economic Growth. New Haven, Yale University Press.

[28] Mangasarian, O. L., 1966. Sufficient Conditions for the Optimal Control of Nonlinear Systems. SIAM Journal of Control 4, 139-152.

[29] Mino, K., 2004. Human capital formation and patterns of growth with multiple equilibria. In: Boldrin,M., Chen, B.-L.,Wang, P. (Eds.), Human Capital, Trade and Public Policy in Rapidly Growing Economies. Edward Elgar, Cheltenham, pp. 4264

[30] Mulligan, C. B., Sala-i-Martin, X.,1993. Transitional dynamics in two-sector models of endogenous growth. The Quaterly Journal of Economics 108, 739-773.

[31] Naz, R., Mahomed, F. M., and Chaudhry, A., 2014. A Partial Hamiltonian Approach for Current Value Hamiltonian Systems. Communication in Non-linear Science and Numerical Simulation 19, 3600-3610.

[32] Park, H., Philippopoulos, A., 2004. Indeterminacy and fiscal policies in a growing economy. J. Econ. Dyn. Control. 28, 645660.

[33] Panayotou, T., 2000. Economic Growth and the Environment. Harvard University and Cyprus International Institute of Management.

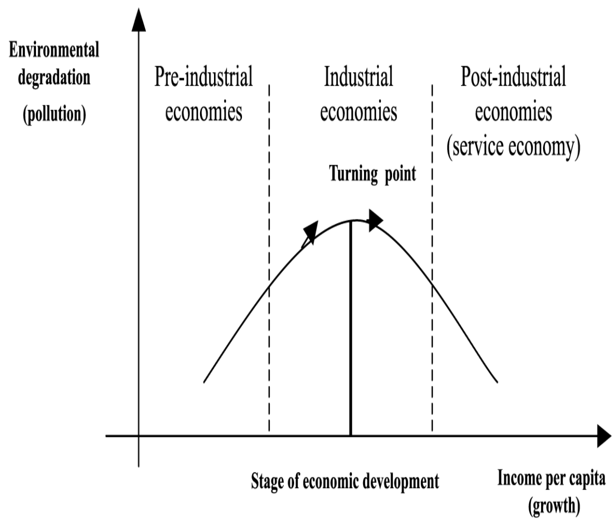
[34] Pérez, R., Ruiz, J., 2007. Global and local indeterminacy and optimal environmental public policies in an economy with public abatement activities. Economic Modelling 24, 431452.

[35] Romer, M. P., 1986. Increasing Returns and Long-Run Growth. Journal of Political Economics 94, 1002-1073.

[36] Smulders, S. A., 1999. Economic Growth and Environmental quality. Priciples of environmental economics, ch 21.

[37] Xie, D., 1994. Divergence in economic performance: transitional dynamics with multiple equilibria. Journal of Economic Theory 63, 97-112.

[38] Yanqing, X. and Mingsheng, X., 2012. A 3E Model on Energy Consumption,Environmental Pollution and Economic Growth—An Empirical Research Based on Panel Data. Journal of Energy Procedia 16, 2011-18.



Source: Panayotou (1993)

Appendix A

Appendix B

The necessary first order conditions for optimal control are

$$\frac{\partial H}{\partial C} = (CE^{\gamma})^{-\sigma}(E^{\gamma}) - \lambda = 0$$

$$\lambda = \frac{(E^{\gamma(1-\sigma)})}{(C^{\sigma})}$$

$$\frac{\partial H}{\partial \lambda} = \dot{K} = (1-\tau)K^{\alpha}E^{1-\alpha} - C$$

$$\frac{\dot{K}}{K} = (1-\tau)K^{\alpha-1}E^{1-\alpha} - \frac{C}{K}$$

$$-\frac{\partial H}{\partial K} + \rho\lambda = \dot{\lambda} = -\lambda\alpha K^{\alpha-1}E^{1-\alpha}(1-\tau) + \rho\lambda$$

$$\frac{\dot{\lambda}}{\lambda} = -\alpha K^{\alpha-1}E^{1-\alpha}(1-\tau) + \rho$$

Differentiating equation (1.13) with respect to time t yields

$$\dot{\lambda} = -\sigma E^{\gamma(1-\sigma)}C^{-\sigma-1}.(\dot{C}) + C^{-\sigma}\gamma(1-\sigma)(E^{\gamma((1-\sigma)-1)})\dot{E}$$

$$\dot{\lambda} = C^{-\sigma}E^{\gamma(1-\sigma)}[-\sigma C^{-1}\dot{C} + \gamma(1-\sigma)E^{-1}\dot{E}]$$

and thus

$$\frac{\dot{\lambda}}{\lambda} = -\sigma\frac{\dot{C}}{C} + \gamma(1-\sigma)\frac{\dot{E}}{E}.$$

The growth rate of consumption is

$$\sigma\frac{\dot{C}}{C} = \gamma(1-\sigma)\frac{\dot{E}}{E} - \frac{\dot{\lambda}}{\lambda}$$

$$\frac{\dot{C}}{C} = \frac{\gamma(1-\sigma)}{\sigma}\frac{\dot{E}}{E} - \frac{\dot{\lambda}}{\lambda\sigma}$$

Substituting the values of $\frac{\dot{E}}{E}$ and $\frac{\dot{\lambda}}{\lambda}$ in the above equation, we have

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}[\gamma(1-\sigma)(T-\theta)(\frac{K}{E})^{\alpha} + (\alpha)(\frac{K}{E})^{\alpha-1}(1-\tau) - \rho].$$

The differential equations representing the dynamics of this model can be summarized as follows:

$$\frac{\dot{K}}{K} = (1-\tau)K^{\alpha-1}E^{1-\alpha} - \frac{C}{K}$$

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}[\gamma(1-\sigma)(T-\theta)(\frac{K}{E})^{\alpha} + (\alpha)(\frac{K}{E})^{\alpha-1}(1-\tau) - \rho].$$

$$\frac{\dot{E}}{E} = (T-\theta)(\frac{K}{E})^{\alpha}$$

Appendix C

Analysis of equilibrium

we will assume that

$$g = \frac{\dot{E}}{E} = (T-\theta)(\frac{K}{E})^{\alpha}$$

similarly

$$z = \frac{K}{E} = (\frac{g}{T-\theta})^{\frac{1}{\alpha}}$$

$$\frac{\dot{K}}{K} = h = (1-\tau)(\frac{K}{E})^{\alpha-1} - \frac{C}{K}$$

$\dot{Z} = 0$ which implies that

$$X^{*} = (1-\tau)Z^{\alpha-1} - (T-\theta)Z^{*}\alpha \geq 0(Z=0, X=0)$$

$\dot{X} = 0$ implies

$$\frac{1}{\sigma}[\gamma(1-\sigma)(T-\theta)Z^{*\alpha} + \alpha(1-\tau)Z^{*\alpha-1} - \rho] - (1-\tau)Z^{*\alpha-1} + (1-\tau)Z^{*\alpha-1} - (T-\theta)Z^{*\alpha} = 0$$

$$\gamma(1-\sigma)(T-\theta)Z^{*\alpha} + \alpha(1-\tau)Z^{*\alpha-1} - \rho - (T-\theta)\sigma Z^{*\alpha} = 0$$

equation (30) can only be true if it satisfies the equation below

$$(T-\theta)Z^{*\alpha}[\gamma - \gamma\sigma - \sigma] + \alpha(1-\tau)Z^{*\alpha-1} - \rho = 0$$

$$\dot{Z} = (1-\tau)Z^{\alpha} - XZ - (T-\theta)Z^{\alpha+1}$$

Appendix D

Stability and transitional dynamics around equilibrium

$$\begin{bmatrix} X(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} X(t) - X^* \\ Z(t) - Z^* \end{bmatrix}. \tag{2.9}$$

$$a_{21} = \frac{\partial \dot{Z}}{\partial X} = -Z^*$$

$$\frac{\partial \dot{Z}}{\partial Z} = \alpha(1-\tau)Z^{*\alpha-1} - X^* - (\alpha+1)(T-\theta)Z^{*\alpha}$$

$$\frac{\partial \dot{Z}}{\partial Z} = \alpha X^* - (T-\theta)Z^{*\alpha} - X^*$$

$$a_{22} = \frac{\partial \dot{Z}}{\partial Z} = (\alpha-1)X^* - (T-\theta)Z^{*\alpha}$$

$$\frac{\partial \dot{X}}{\partial X} = \frac{1}{\sigma}[\gamma(1-\sigma)(T-\theta)Z^{\alpha} + \alpha Z^{\alpha-1}(1-\tau) - \rho - (1-\tau)Z^{\alpha-1}\sigma] + 2X$$

$$a_{11} = \frac{\partial \dot{X}}{\partial X} = -X^* + 2X^* = X^*$$

$$\frac{\partial \dot{X}}{\partial Z} = \frac{X}{\sigma}[\gamma(1-\sigma)(T-\theta)\alpha Z^{\alpha-1} + \alpha(\alpha-1)(1-\tau)Z^{\alpha-2}] - (1-\tau)X(\alpha-1)Z^{\alpha-2}$$

$$\frac{\partial \dot{X}}{\partial Z} = \frac{X}{\sigma}Z^{\alpha-2}[\gamma(1-\sigma)(T-\theta)\alpha Z + (\alpha-1)(1-\tau) - \sigma(1-\tau)(\alpha-1)]$$

$$a_{12} = \frac{\partial \dot{X}}{\partial Z} = \frac{X}{\sigma}Z^{\alpha-2}[\gamma(1-\sigma)(T-\theta)\alpha Z + (\alpha-1)(1-\tau)(\alpha-\sigma)]$$

Since this linearized system include one control(jump) variable(X) and one state variable Z, there are three possibilities for the local dynamics (1) if $det < 0$ than it will be locally saddle path stable, (2) if $det > 0$ and $trace > 0$ than it will be locally unstable and (3) if $det > 0$ and $trace < 0$ than it will be locally indeterminate. So we will calculate the determinate of 2×2 matrices.

$$detJ^* = X^*[(\alpha-1)X^* - (T-\theta)Z^{*\alpha}] + \frac{X}{\sigma}Z^{\alpha-1}[\gamma(1-\sigma)(T-\theta)\alpha Z + (\alpha-1)(1-\tau)(\alpha-\sigma)]$$

Since the sign of the determinant is difficult to identify we will follow the idea Hosoya (2012) and assume that $\alpha = \sigma$ which implies that the share of the physical capital in the production function and the inverse of the intertemporal elasticity od substitution have the same value in the utility function. Now after applying specific assumption to the determinant leads to

$$detJ^* = (\sigma - 1)X^{*2} - (T - \theta)X^*Z^{*\sigma}[1 + \gamma\sigma - \gamma]$$

In the equation above $X^* > 0$ and $Z^* > 0$. Since $\gamma \geq 0$ and $\sigma \in (0, 1)$, by considering suitable considerations for the parameters σ and γ we can confirm that $detJ < 0$ means respective model is saddle path stable.

$$\frac{\partial \dot{E}}{E} = (T - \theta)(\frac{K}{E})^\alpha = g, whereas(g = \frac{\dot{E}}{E} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C})$$

$$(\frac{K}{E})^\alpha = \frac{g}{T - \theta}$$

$$\frac{K}{E} = (\frac{g}{T - \theta})^{\frac{1}{\alpha}} = Z$$

$$g = (1 - \tau)(\frac{g}{T - \theta})^{\frac{\alpha - 1}{\alpha}} - \frac{C}{K}$$

$$\frac{C}{K} = X = (1 - \tau)(\frac{g}{T - \theta})^{\alpha - 1}\alpha - g$$

$$\dot{g} = \frac{1}{\sigma}[\gamma(1 - \sigma)(T - \theta)\frac{g}{T - \theta} + (1 - \tau)(\frac{g}{T - \theta})^{\frac{\alpha - 1}{\alpha}} - \rho]$$

$$g\sigma = \gamma(1 - \sigma)(T - \theta)\frac{g}{T - \theta} + \alpha(1 - \tau)(\frac{g}{T - \theta}^{\frac{\alpha - 1}{\alpha}} - \rho$$

$$\alpha(1 - \tau)(\frac{g}{T - \theta})^{\frac{\alpha - 1}{\alpha}} = [\sigma - (1 - \sigma)\gamma]g + \rho$$

Appendix E

Stability of equilibrium by eigenvalues

For a 2×2 system the stability can be analyzed by the signs of determinant and trace of the Jacobian matrix. There are three possibilities: (a) det< 0 (saddle-path stable); (b) det> 0 and tr> 0 (unstable) and (c)det> 0 and tr< 0 (stable). We analysis stability of equilibrium by eigenvalues as follows:

	Eigen value type	stability
Case I:	Both are real and positive	Unstable node
	Both are real and negative	Stable node
	Real: one positive and one negative	Saddle path stable
Case II:	Both are equal, real and positive	Unstable node
	Both are equal, real and negative	Stable node
Case III:	Complex with positive real parts	Unstable focus
	Complex with negative real parts	Stable focus
	Complex with zero real part	Unstable