Endogenous time preferences and Environmental quality: Multiple equilibria and Fiscal implications

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ABSTRACT

This thesis analyzes a model of growth with endogenous time preferences, taking forward the model by Chaudhry et al. (2017). Economic growth is determined by factors like consumption, income, savings etc., and this thesis will include environmental quality as a factor in the production function. The analysis will focus on how incorporating endogenous discounting into individual preferences affects economic growth. I hypothesize the presence of multiple equilibria, one low growth and one high growth. Also the requisite stability analysis will be done to look at stable equilibria along the balanced growth path. Furthermore, I will analyze how fiscal policy can be used to affect the growth rate in the economy.

DECLARATION

I hereby declare that this is my own work. It is being submitted for the Degree of M.Phil. in Economics to the Lahore School of Economics, Pakistan. It has not been submitted before for any degree or examination to any other University.

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INTRODUCTION

Economic growth as a subject has been studied in detail. The literature has vast theories on how growth is affected. Several factors are listed that affect growth of an economy and the list includes but is not limited to level of output, income, individual's utility (Yanase, 2011) and welfare in an economy. Gupta and Barman (2010) attribute economic growth to positive and negative externalities; positive externalities like health, education, infrastructure etc. increase growth and the negative externalities like pollution, environmental quality, congestion etc. retard growth.

Growth theories on what affects economic growth vary from classical growth models, where an invisible hand guides the growth in any economy through self regulated and efficient markets, and growth results when overall inputs (namely land, labor and capital) are increased (Smith, 1980, 1976a,b, 1978, 1983); to neoclassical models (Lucas, 1988; Solow, 1956; Koopmans, 1965) which theorize that certain factors (health, education, etc.) in the economy actually give the necessary jump to the economy in order to grow.

Education is also a factor that affects economic growth. Barro (2013) looks empirically how education affects growth; education increases the level of learning and adopting new technology, which will lead to growth.

Many studies have been undertaken, to look at growth from the perspective of health. Agénor (2008) looks at the effect of health on economic growth. **Similarly Zhang et al. (2003) note that a healthy individual will save**

more, resulting in more capital accumulation and hence growth will increase. In Agénor (2010), the change in health due to infrastructure, is examined. **Nelson and Phelps (1966) theorize that the stock of health will trigger the growth in the economy in terms of helping through technological innovation.**

Cutler and Musney (2012) give an overview of a relationship between education and health, showing that some relation exists between both poor countries. Bobonis et al. (2006) prove empirically that a healthy individual will have better attendance in school; similarly Maluccio et al. (2009) also focus on how better health can lead to increased education and hence earnings.

However, modern economic growth theories look at other factors that affect development and economic growth; the environment's affect on growth is a relatively newer field and is also a part of modern economic growth theory. The current literature has focused heavily on research on environmental quality and its effect on economic growth. In fact, the recent literature looks at the environmental Kuznets Curve (EKC), which explains the relationship between growth and environment (its betterment or degradation). The EKC shows an inverted U-shape relationship between economic growth and pollution; initially there is environmental degradation as the economy grows, and eventually there is improvement. (See Appendix A)

As Chaudhry et al. (2017) find, this relationship is important for growing economies, where increasing production and consumption results in the depletion of a country's resources-both environmental and otherwise. The relationship can change according to the level of income in the economy; meaning that individual's can demand better economic quality (Panayotou, 2000). Dinda (2005) then explains that initially the economies do not invest in pollution abatement leading to environmental degradation, but as economy grows investment in abatement results in a better environmental quality.

Grossman and Krueger (1995) study the relationship between economic activity and environmental quality-through studying a number of environmental indicators; they empirically prove that there is environmental degradation as the economy grows, using the Global Environmental Monitoring System's (GEMS) panel data. This is then followed by improvement in the environmental quality as further growth takes place. Similarly, Millimet et al. (2003) also empirically examine the Kuznets curve relationship and find that it holds for US states. For this thesis, time-preferences are endogenized and are dependent on the consumption of an individual and the environmental quality that an individual experiences. Decisions of an individual are hence based on current and future consumption, while environmental quality changes exogenously.

In addition to all the above relationships, there is an increasing understanding that time preferences of individuals also affect the economic growth; be it with respect to consumption, health, education, patience, saving, investment, etc. (Becker and Mulligan, 1997). Kawagishi (2012) looks at how investment decisions affect time preferences i.e. the discount rate.

Today, endogenous time preferences are important for modern economic growth theories. Research on this help understand long-run relationships and effects of any policy; also observe the intergenerational consumption, investment, saving etc. patterns for effective policy making. Factoring in the environmental quality enables us to see how the all growth factors are effected with respect to environmental conditions. (Hepburn et al., 2009; Freeman et al., 2015).

Literature on how time preferences are affected by environment, however is a newer subject and is less researched. There is empirical evidence that links environment and the patience of an individual, in essence affecting economic growth. The link that relates to both environment and patience is the health or human capital of an individual. Health is impacted through air pollution, resulting in workers being absent and or being unproductive (Graff Zivin and

Neidell, 2013; Hanna and Oliva, 2015; Graff Zivin and Neidell, 2012). For these studies, health is a function of ambient pollution levels (Graff Zivin and Neidell, 2013; Cropper and Freeman, 1991; Graff Zivin and Neidell, 2012). The negative effect of this pollution is either overcome by avoidance behavior, or through the medical care; both ways help overcome the negative consequences from pollution exposure (Graff Zivin and Neidell, 2012; Cropper and Freeman, 1991; Graff Zivin and Neidell, 2013). Aside from this contemporaneous effect on human capital, there are studies which show that latent affects are also playing a role. Pollution or environmental quality is found to have pure biological effects, namely effecting skills formation (Cunha and Heckman, 2007; Almond and Currie, 2011). Effects have also been seen on student performance at the high school level, if the mother was exposed to pollutants prenatally (Sanders, 2012).

Hence effect of environmental quality affects health and human capital, which can span an individual's lifetime. In this context, individuals take life decisions and make choices regarding consumption and investment etc. This gives rise to the life history theory whereby human behavior is explained; especially if an individual adopt to a changing environment faster or slower. The life history theory helps an individual take decisions with respect to reproductive decisions; and also ecological factors that help improve their health (Belsky et al., 1991; Del Giudice, 2009). In the context of this thesis, the life history strategies used will be those where an individual makes a decision regarding current and future investment (Griskevicius et al., 2011).

An experiment that Griskevicius et al. (2011) performed showed that mortality cues affected life history strategies, and that these cues were taken from a person's childhood socioeconomic statuses (SES). The study showed that a higher SES resulted in better health, making individuals patient. Furthermore, individuals with lower childhood SES, preferred the current time period more (Griskevicius et al., 2011). Green et al. (1999) note that discounting increases with age; people grow more patient as they age.

Empirical literature provides us with how environmental quality affects health of an individual, which in turn affect the preferences of an individual, in how he prefers to consume, invest, save, etc. These set of preferences then affect the growth. Better environmental quality ensures a patient individual, who prefers to consume less today (healthy individual, life span increases) and save for the future, ensuring more investment and hence more growth (Green et al., 1999; Graff Zivin and Neidell, 2013).

The literature above has provided a number of factors that affect economic growth, especially the environmental quality and the preferences of an individual. Hence, in these contexts, there can be a possibility that multiple equilibria exist. In Gaspar et al. (2014) we see that public health infrastructure affects economic growth, resulting in multiple equilibria. Hosoya (2012) similarly establishes multiple equilibria in a model of economic growth and public infrastructure.

The literature indicates that the multiple equilibria can be either a low environmental quality and low growth equilibria or a high environmental quality and a high growth equilibrium (Chaudhry et al., 2017). Lower environmental quality makes the individual impatient, through low quality of health and a limited life span, thus the individual reduces investment and prioritizes current consumption, and hence a low growth equilibrium will be reached (Green et al., 1999; Graff Zivin and Neidell, 2013).

On the other hand, a better environmental quality assures better health and a better future expectation of a healthy life; helping the individual, choose to invest more in the future time period. Hence individual is more patient as the quality of environment gets better (Green et al., 1999; Graff Zivin and Neidell, 2013).

In the above stated scenarios of multiple equilibria, government intervention can help play a role. The government can intervene through its fiscal policies and can help lead to a higher growth equilibrium, with better environmental quality. Endogenous time preferences, an individual's preferences with respect to decisions on consumption today and in the future, will affect growth, depending on the environmental quality present in the economy. This will mean that depending on the environmental quality in the economy, an individual will decide how much to consume today and how much to invest in future. There might also be a possibility that for both high and low environmental quality, a low and a high growth equilibrium exist. In this case as well, there is a role for the government to play. For high environmental quality, government intervention can decrease the gap between high and low growth equilibria; and vice versa for lower quality of environment (Dioikitopoulos and Kalyvitis, 2015; Vella et al., 2015; Chaudhry et al., 2017).

Essentially, countries, with the similar conditions (health levels, patience levels, environment quality, etc.) can end up in a so called bad equilibrium, which has high impatience, poor environmental conditions, and low growth. Alternatively, the same can also end up in a so called good equilibrium, categorized by better environmental quality, lower impatience, and thus higher growth. Thus, Fiscal policy can in the context of such multiple equilibria can help lead to a higher growth equilibrium (Dioikitopoulos and Kalyvitis, 2015; Vella et al., 2015; Chaudhry et al., 2017).

Empirical literature aside, theoretical papers also support a relationship between environment and time preference, which in turn effects economic growth. Le Kama and Schubert (2007) examine this relationship, noting that the environmental quality is positively effecting the discount rate. Showing that in the presence of environmental quality the steady state is reached for only some certain values of parameters, when discounting is taken as exogenous, but steady state reaches generally when there is endogenous discounting (Le Kama and

Schubert, 2007).

Yanase (2011) also looks at this relationship of the discount rate as dependent on individual utility and consumption. The assumption that Yanase (2011) works on is that better environmental quality will make individuals more patient. Multiple steady states may exist in this case, depending on the relationships taken, that is either the pollution-capital relationship taken in the production function, or the pollution-consumption relationship taken under consideration by the individual's utility function; this will all be dependent upon properties of the discount rate or time preference function (Yanase, 2011).

Leach (2008) measure some pros and cons of climate change policies, showing their outcomes as dependent on the discounting that is associated with the policy. Simulations are undertaken to show the results, which show that time preferences matter in decisions with respect to the climate change and the policies (Leach, 2008).

Vella et al. (2015) looks at how environmental quality affects individual's patience, showing that high environmental quality inspires patience. The environmental quality effects indirectly through making an individual more patient, which induces them to save more and hence capital is accumulated. Multiple equilibria exist, which are also explained through government's policies (Vella et al., 2015).

Similarly Dioikitopoulos et al. (2016) also look at how endogenous discounting effects growth in an economy, when there is technological progress, and how environmental quality and its changes affect the equilibrium.

Varvarigos (2013) similarly examines the relationship between pollution and capital accumulation, showing that as pollution rises life expectancy falls and saving falls so there is less capital accumulation. The study finds a unique steady state equilibrium but shows that if abatement technology is taken as endogenous to the model along with a tax on emissions, multiple equilibria may exist. Kuznets curve holds in presence of endogenous government policy (Varvarigos, 2013).

Most studies also show that there exist multiple equilibria when solving an endogenous time preference model, which takes environmental quality as exogenous. So here we can see that some equilibria will have a positive relationship with environmental quality while the others will lead to low environmental quality and low growth traps (Vella et al., 2015).

Additionally, the presence of multiple steady states or equilibria exists in literature. Hosoya (2012) notes that there exist two equilibria, namely a high and low growth equilibrium, along the balanced growth path (BGP). Mino (2003) also looks at how in a model including human capital, can lead to multiple equilibria, showing variation in the economic growth of different countries.

Gaspar et al. (2014) also establishes that there exist multiple equilibria in theory and literature, showing that most economic growth is encompassed by these multiple growth paths. Similarly Park and Philippopoulos (2004) also look at multiple equilibria, using capital taxes into its model with endogenous labor supply. Both studies show that these multiple growth paths can either be stable, unstable, or undetermined.

These equilibria are further affected by fiscal decisions and policies, especially in light of this thesis. Agénor (2010) looks at how the government's spending on public infrastructure affects economic growth; an efficient system of government investment in the public infrastructure, results in a shift away from the low equilibrium.

Agénor (2008) studies how government allocation to infrastructure and health can affect growth. Their analysis is twofold, one where health is a variable in the production and utility function, and the other where health itself is a stock variable (Agénor, 2008). They consider that the infrastructure affects the production of goods and also the health services (Agénor, 2008).

Barro (2013) indicates that educational attainment, that is an increase in human capital results in increased growth. Public policies to promote human capital will result in the above mentioned result (Barro, 2013).

This thesis focuses on how fiscal policies can affect the multiple equilibria obtained. Dioikitopoulos et al. (2016) also look at how the equilibria are affected by economic, mainly fiscal, policy. In another paper Dioikitopoulos and Kalyvitis (2015) looks at how time preferences are dependent on consumption and human capital stock; they also look at optimal fiscal policy similar to that of Ramsey taxation.

Economides and Philippopoulos (2008) look at a Ramsey-type government tax, a distortionary tax that finances infrastructure for environmental cleanup. This paper notes that such a policy will be growth enhancing without damaging the environment. This policy comes out to be optimal in cases where the household is more careful about the environment; and also, the resources are allocated as such to protect the environment (Economides and Philippopoulos, 2008).

Gupta and Barman (2010) takes forward the model by Agénor (2008) by including in environmental pollution and in addition analyzes this in light of the government tax revenues. As Chaudhry et al. (2017) note, government expenditures should also focus on abatement activities so that the environmental quality is better and hence the economic growth is better. Literature has established the existence of these multiple equilibria, and noted that fiscal policies and implications can help eliminate or change the low equilibrium path to a better equilibrium path.

In this light, this thesis will examine an endogenous growth model similar to Chaudhry et al. (2017), that is observing effect of growth on the environmental quality, but taking time preferences as exogenous. The difference in the current thesis would be that I will be taking time preferences as endogenous, and looking at the growth relationship with respect to how discounting will be effected by consumption, fiscal policies, and environmental quality. The relationship is made through its effect on an individual's health and their life history theories.

Furthermore, this thesis will differ from Dioikitopoulos et al. (2016) in terms of fiscal implications; this thesis will look at growth enhancing policies instead of an environmental distortionary tax of the form of Ramsey taxation. The Barro (1990) model shows that the optimal marginal tax rate is given by $\tau = (1 - \alpha)$where the rate of time preferences are constant (ρ'(.) $= 0$). Growth enhancing fiscal policies or tax rates will be either higher or lower when rate of change of time preferences is not taken as constant (ρ'(.) $\neq 0$) (Dioikitopoulos et al., 2016).

This thesis will contribute to literature in a number of ways. A model will be defined which uses endogenous time preferences in the utility and production functions. The model incorporates the endogenous time preferences in which the rate of time preference depends positively on the ratio of aggregate consumption and environmental quality, where both of these factors are determined exogenously (through life span theories of Griskevicius et al. (2011)). Furthermore, the stability analysis will also be done for these equilibria. Lastly the implications of this model will be seen with respect to some fiscal implications.

The model will be solved in the standard mathematical procedures for BGP and steady state values, as is done in literature as well. Here unique and multiple equilibria are predicted. Hence stability analysis will be needed to go forward. This will be similar to Hosoya (2012), Dinda (2005), and also Benhabib and Perli (1993). Finally, an endogenous fiscal policy will be found.

The model itself is explained in Chapter 1. Chapter 2 looks at the expected outcomes of the model and the BGPs stability analysis, and Chapter 3 will then look at the finding a growth maximizing fiscal policy for the model. Finally

Chapter 4 will conclude the thesis.

Chapter 1

The model incorporating time preferences and environment

In this chapter we take forward the idea of Chaudhry et al. (2017), a model with environment, and incorporate into it endogenous time preferences. We will explain the framework of the model and how the time preferences effect growth in presence of environmental quality.

1.1 Framework of the model

As seen through literature, economic growth is affected by and also affects the environmental quality of a country. Numerous studies have seen this effect. I examine the same effect but take into account time preferences (endogenous to the model). In addition, I also measure changes in the above relationship if fiscal implications are present.

This relationship will rest on the premise that environment effects the health of an individual and that in turn effects their discount rates through decisions based on their life history theories (Graff Zivin and Neidell, 2013; Griskevicius et al., 2011).

I use a present value Hamiltonian, to maximize utility, in order to see how time preferences, effect the growth. Here for this model time preferences depend on consumption and the environmental quality over time. I propose that this will give us multiple equilibria.

1.1.1 Production

The model is characterized by a Cobb-Douglas production function, incorporating capital (K) and environmental quality (E), given as under:

$Y(t)$:

$$Y(t) = K(t)^{\alpha}[L(t)E(t)]^{1-\alpha}, 0 < \alpha < 1. \tag{1.1}$$

Here $K(t)$ represents the physical capital in the economy, $L(t)$ represents the total labor force in the economy, and $E(t)$ represents the total stock of environment in the economy; thus making $Y(t)$ the total output in the economy. **$E(t)$, in the production function, and it measures the productivity effect of environmental quality, as it is an input of the production process. Also $E(t)$ is taken as given by the individual and is defined exogenously in the next section.**

This effectively increases the labour productivity as does the human capital in models similar to Lucas' model, which give emphasis to the accumulation of human capital (Lucas, 1988; Caballé, 1993).

The assumption here is that there is no population growth, thus we normalize labour, i.e. $L(t) = 1$ (thus all other variables are in per capita amounts) (Dinda, 2005; Dioikitopoulos and Kalyvitis, 2015; Chaudhry et al., 2017).

However, $E(t)$ is not taken as the choice variable for the agents, on the assumption that the individual agents will view the stock of environment as given for their own production function (Dinda, 2005; Chaudhry et al., 2017). Hence $E(t)$ is exogenous for individuals (Dioikitopoulos and Kalyvitis, 2015; Chaudhry et al., 2017).

Similar to Chaudhry et al. (2017), I take α as the weight given to the physical capital. This, in literature, helps understand the relationship of output to input (physical capital or environmental quality), keeping all other variables equal (Chaudhry et al., 2017).

1.1.2 Law of motion of physical capital

I am assuming that the law of motion for the physical capital is given by, similar to Chaudhry et al. (2017):

$$\dot{K} = I_K. \tag{1.2}$$

Here I_K is the investment into physical capital. Also I_E is the investment into the environmental quality, where:

$$I_E = \tau Y. \tag{1.3}$$

Here τ is the tax rate on abatement (Chaudhry et al., 2017; Hosoya, 2014, 2012, 2017).

The resource constraint is thus as follows:

$$Y = C + I_K + I_E. \tag{1.4}$$

Hence the equation of motion for physical capital comes out to be:

$$\dot{K} = (1 - \tau)Y - C. \tag{1.5}$$

Finally the equation of motion for physical capital evolves to:

$$\dot{K} = (1 - \tau)K^{\alpha}E^{1-\alpha} - C. \tag{1.6}$$

1.1.3 Environment

E is the environmental quality of the economy for this model. The household will maximize utility by taking environmental quality as exogenous. As Chaudhry et al. (2017) and Gupta and Barman (2010) note, the environmental quality will be measured by an equation of motion, because change in environmental quality over time is seen through the amount of pollution there is in the economy and what is being done for its reduction.

The function thus is given by:

$$\dot{E} = (\tau - \theta)K^{\alpha}E^{1-\alpha},\ 0 < \tau < 1,\ 0 < \theta < 1, \tag{1.7}$$

where θ is the level of emissions effecting production directly; and τ is the ratio of abatement expenditure to national income (Chaudhry et al., 2017; Gupta and Barman, 2010).

Environmental quality changes with respect to use of natural resources and other environmental indicators. So this equation of motion indicates that the quality of environment changes over time due to waste materials categorized under the emissions; and there is an improvement in the quality as abatement action is undertaken. Environmental quality falls because the environment's ability to cater to the emissions is finished, as a result abatement costs can become high and hence investment into its improvement will be lower (Brock and Tylor, 2004). This also indicates that production of physical capital creates emissions and abatement through technological progress brings emissions down (Brock and Tylor, 2004).

However, in this model there is a balanced budget, indicating that revenue from the tax (on abatement of pollution etc.) is spent on the abatement expenditure. It is important to note that this amount of expenditure can be reached when the economy is at a certain level of accumulation of physical capital (Dinda, 2005).

Literature has established a relationship between economic growth and environmental quality, the EKC (see Appendix A). In this context, a country will have to allocate funds for abatement activities to improve environmental quality; the inverted u-shape will emerge (Kuznets, 1966; Stern et al., 1996)) This model, however, is focusing on developed economies, and thus explains the second half of the EKC, where there is a positive relationship between pollutants and emissions, and economic growth (Chaudhry et al., 2017).

1.1.4 Household

Here, I consider an economy with infinitely lived agents. The population growth rate is zero, thus a constant population for this model, consuming a single good. Extending the model of Chaudhry et al. (2017) and in line with Dioikitopoulos and Kalyvitis (2010) the optimization problem for the household is given by:

$$Max \quad \int_0^{\infty} \frac{(C^{\nu} E^{\gamma})^{1-\sigma} - 1}{1-\sigma} e^{\int -\rho(C,E)dt}, \sigma > 0, \sigma \neq 1, \gamma \geq 0, \rho > 0, \tag{1.8}$$

subject to

$$\dot{K} = (1-\tau)K^{\alpha}E^{1-\alpha} - C, \; 0 < \tau < 1, 0 < \theta < 1, \tag{1.9}$$

where ν is the weight given to consumption and γ is the weight given to environmental quality. This is a continuous time growth model, discounting is done in the household utility function, taking the discount rate ρ (which is a function of C, E). ρ shows the patience of an individual taken as a function of C (Consumption) and E (Environmental Quality); a positive value showing future preference. ρ is an endogenously determined function of the ratio of C and E. We will consider linear impatience function as shown by Green et al.

(1999). The solution to this model is given in the next section and solved in Appendix B.

The assumptions, for time discounting, are in line with literature from numerous resources like Choi et al. (2008), Agénor (2010), Dioikitopoulos and Kalyvitis (2010), Dioikitopoulos and Kalyvitis (2015), Vella et al. (2015), Dioikitopoulos et al. (2016), Palivos et al. (1997), Meng (2006), etc.

These assumptions are given as follows:

Assumption 1:

$$\rho(C, E) \geq \breve{\rho} > 0. \tag{1.10}$$

Assumption 2:

$$\rho(C, E) = \rho(C/E),\ \rho'(.) >= 0. \tag{1.11}$$

Assumption 3:

$$\frac{\partial \rho}{\partial C} \geq 0 \tag{1.12}$$

$$\frac{\partial \rho}{\partial E} \leq 0 \tag{1.13}$$

Here assumption 1 shows that there is a lower bound for ρ and that $\breve{\rho}$ is strictly positive. Assumption 2 shows that the preferences are determined by the ratio of consumption and environmental quality; showing that the rate of time preferences is bounded at the steady state (Palivos et al., 1997; Meng, 2006); (Dioikitopoulos and Kalyvitis, 2010). Assumption 3 is in line with the literature, indicating that the change in time preferences with respect to aggregate consumption is positive and the change in time preferences with respect to environmental quality is negative (Dioikitopoulos and Kalyvitis, 2010). This indicates that the higher the aggregate consumption in the economy the lower is the patience of an individual (Choi et al., 2008). This might be the result of individuals becoming richer, as aggregate consumption increases, and in "keeping up with the Joneses" they consume more in the current time period (Dioikitopoulos et al., 2016). Also, assumption 3 shows that an individual is more patient as the environmental quality increases as seen in Dioikitopoulos and Kalyvitis (2010) and Dioikitopoulos and Kalyvitis (2015). A similar theory was presented by Yanase (2011), that life expectancy decreases as pollution increases. The individual will consume less in current time period if the environmental quality is good as they feel more secure about the future. Similarly, if the environmental quality is low then there is a higher value of time preferences, and consumption will be more in the current period. Hence according to assumption 3, ρ is a positive function of the ratio of consumption to environmental quality.

σ is the inverse of intertemporal elasticity of substitution, showing how the individuals will consume as per their time preferences (Chaudhry et al., 2017).

1.2 Solution to the model

We are assuming infinitely lived identical agents in this model. As stated before we are taking forward the model of Chaudhry et al. (2017) and keeping it in line with Dioikitopoulos and Kalyvitis (2010). The sections ahead look at the solution to the model.

1.2.1 Solving the model

The present value Hamiltonian for the model is:

$$H = \frac{(C^{\nu}E^{\gamma})^{1-\sigma} - 1}{1-\sigma}e^{-\Delta(t)} + \lambda[(1-\tau)K^{\alpha}E^{1-\alpha} - C], \tag{1.14}$$

where $\Delta(t) = \int \rho(C, E)dt$, $C(t)$ is the control variable, $K(t)$ is state variable that can not be controlled but is affected by the control variable, $\lambda(t)$ denotes costate variable. I provide the solution of the dynamic optimization problem in Appendix B, finally arriving at following system of two ordinary differential equations:

$$\frac{\dot{K}}{K} = (1-\tau)\left(\frac{K}{E}\right)^{\alpha-1} - \left(\frac{C}{K}\right), \tag{1.15}$$

$$\frac{\dot{C}}{C} = \frac{1}{\nu(1-\sigma)-1}\Bigg[-\gamma(1-\sigma)(\tau-\theta)(\frac{K}{E})^{\alpha} - \alpha(1-\tau)(\frac{K}{E})^{\alpha-1} + \rho(\frac{C}{E})\Bigg]. \tag{1.16}$$

The joint concavity condition of utility function (in C and E) is not required as E is exogenous variable, and it is determined from the following equation:

$$\frac{\dot{E}}{E} = (\tau-\theta)\left(\frac{K}{E}\right)^{\alpha}. \tag{1.17}$$

The utility function is concave in consumption C and constraint is concave in C and K, hence the Mangasarian (1966) sufficiency theorem conditions are met.

In order to solve this system of equations, we introduce two new ratios of variables, and these are given as $X = C/K$ and $Z = K/E$; where X is the control variable while Z is the state variable. Hence the above system of differential equations i.e. (1.15)-(1.17) transforms to the following system of ordinary differential equations:

$$\frac{\dot{X}}{X} = \frac{1}{\nu(1-\sigma)-1}[-\gamma(1-\sigma)(\tau-\theta)Z^{\alpha} - \alpha(1-\tau)Z^{\alpha-1} + \rho(XZ)] - (1-\tau)Z^{\alpha-1} + X \tag{1.18}$$

$$\frac{\dot{Z}}{Z} = (1-\tau)Z^{\alpha-1} - X - (\tau-\theta)Z^{\alpha} \tag{1.19}$$

This system of equations will be solved along the balanced growth path (BGP) to determine the steady state in the next section.

1.2.2 The equilibrium along the BGP

This section shows the solution to the model. Since at the steady state all variables grow at the same constant growth rate, X and Z become constants. Hence the growth rate g is thus as follows:

$g = \frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{E}}{E} = \frac{\dot{Y}}{Y}, X = X^*, Z = Z^*.$

For this model, we find the values for X^* and Z^* only, where:

$$X^* = (1-\tau)Z^{*\alpha-1} - (\tau-\theta)Z^{*\alpha}, \tag{1.20}$$

and Z^* satisfies the following equation:

$$\frac{1}{\nu(1-\sigma)-1}[-\gamma(1-\sigma)(\tau-\theta)Z^{*\alpha} - \alpha(1-\tau)Z^{*\alpha-1} + \rho(X^*Z^*)] - (\tau-\theta)Z^{*\alpha} = 0. \tag{1.21}$$

And ρ is a function of X and Z; which will later be specifically defined as:

$$\hat{\rho} = \beta(X^*Z^*) + \breve{\rho}, \tag{1.22}$$

where $X^*Z^* = C^*/E^*$

We can furthermore see that we have two functions satisfied by Z, namely $\Gamma(Z)$ and $\Psi(Z)$

$\Gamma(Z) = \frac{\alpha(1-\tau)Z^{\alpha-1}}{1-\nu(1-\sigma)} - (\tau-\theta)Z^{\alpha} + \frac{\gamma(1-\sigma)(\tau-\theta)Z^{\alpha}}{1-\nu(1-\sigma)},$

$$\Psi(Z) = \frac{\rho(XZ)}{1-\nu(1-\sigma)} \ .$$

These will help in solution through numerical simulations. The solution will require simulations for possibility of BGPs.

1.3 Numerical simulations

In this section, numerical simulations are undertaken to explore the theoretical basis of the model and its solutions. The simulations can help provide a clearer picture of the unique or multiple BGPs, which are discussed in the previous section.

The literature determines the parameters values that can be used in the solution to the model.

For values of ρ, the discount rate, Hosoya (2012) uses a $\rho = 0.05$, while Antoci et al. (2011) uses a very low discount rate, i.e. 0.001; authors like Itaya (2008) use 0.045 as the discount rate for their solution. Hosoya (2014) uses $\rho = 0.05$ for a unique equilibrium and $\rho = 0.1$ for the multiple equilibrium case. Similarly, Hosoya (2017) once again uses a discount factor of 0.1 in the article to assess the steady states; Chaudhry et al. (2017) also then takes this discount factor forward for their analysis. Dioikitopoulos and Kalyvitis (2015) uses a discount rate of 0.005 for their analysis. Authors like Ladrón-de-Guevara (1997) and Carboni et al. (2013) use the discount factor in a range of 0.05 to 0.08.

We will now define the functional form of the Impatience Function. It is defined similar to the functional form from Dioikitopoulos and Kalyvitis (2015); thus we define the Impatience Function for this model as follows:

$$\hat{\rho} = \beta(C/E) + \breve{\rho}. \tag{1.23}$$

In addition, the values of σ, the inverse of intertemporal elasticity of substitution, is varied in the literature, as some take a higher value for it and some take a lower values for it. For Carboni et al. (2013), the value is 0.01, for Hosoya (2012) it is 0.35, while for Ladrón-de-Guevara (1997) it is 0.7. These are the lower values for σ. Hosoya (2017) uses a higher value, i.e. 3.8. Among the high values are also: Itaya (2008), Antoci et al. (2011), and Hosoya (2014), as 2, 1.5, and 1.2 respectively.

The parameter, γ, is the weight given to the environmental quality in the production function. This parameter is important for the analysis of this model. Itaya (2008) uses a value of 1, and Fernández et al. (2012) uses 4.60. Environmental quality is used by the authors Nguyen-Van and Pham (2013) in the utility function; they use $\gamma = 0.2$ as the weight on environmental quality. Chaudhry et al. (2017) also use environmental quality in their analysis and the values are from 0.54 to 3.61 for a unique equilibrium, 3.62 to 4.25 for two equilibria, and greater than 4.25 for no solution to exist.

In the literature, τ and θ, the tax rate on abatement and emission-output coefficient respectively, are taken as follows: Chaudhry et al. (2017) use a tax rate of 0.05, similar to that of Hosoya (2014). In Hosoya (2012), a tax rate of 0.04 is used, while Hosoya (2017) uses a tax rate of 0.025. Dioikitopoulos and Kalyvitis (2015) uses a tax rate of 0.4 for the analysis.

Similarly, the value of θ varies and Chaudhry et al. (2017) use a value of 0.042, Hosoya (2012) uses a value of 0.35 for the analysis and repeats the same value in Hosoya (2017). Most authors find that the emissions, especially the carbon monoxide and sulphur dioxide emissions increase as the income of low income countries increase; making θ positive (Shafik and Bandyopadhyay, 1992; Grossman, 1995; Seldon and Song, 1994).

For this section, and the numerical simulation analysis, the following values for the parameters are taken: $\alpha = 0.35$, $\beta = 0.1$, $\breve{\rho} = 0.001$, $\sigma = 0.5$, $\tau = 0.022$, $\theta = 0.017$, $\gamma = 1.5$, $\nu = 1$. These are the benchmark values from among the range of values which are seen in the literature, that the solution to the model will be interpreted on. The table (1.1) shows the benchmark parameter values that I will be using for the simulations' sections ahead.

Table 1.1: Parameters-Benchmark Values

γ	σ	β	α	$\breve{\rho}$	τ	θ	ν
1.5	0.5	0.1	0.35	0.001	0.022	0.017	1

Through the simulations, we study the resulting equilibria when values of these parameters are changed, in a manner similar to Chaudhry et al. (2017). In particular we change the value of γ, the weight of the environmental quality, keeping all other parameters fixed. For this model, under the benchmark values provided in table 1.1, a unique equilibrium exists for $0 \leq \gamma \leq 0.36$; we get multiple equilibria for all values of $\gamma > 0.36$.

1.4 Multiple equilibria and changes in growth rate

From the last section we have found that for multiple (two) equilibria to exist $\gamma > 0.36$. Now simulations are undertaken to observe how changes in the weight of the environmental quality, γ, the inverse of intertemporal elasticity of substitution, σ, and the slope of the impatience function, β, affect the growth rates, and other pertinent variables.

Table 1.2: Effect of change in parameters

		g_L	g_H
		0.0078	0.0314
γ	2.5	0.0079	0.0311
σ	0.4	0.0079	0.0312
β	0.15	0.0068	0.0315

The high and low equilibrium growth rates are defined as g_H and g_L respectively (similar to Chaudhry et al. (2017)). Changes in these growth rates are then compared with the benchmark values given in table 1.1.

The table 1.2 observes the effect of changes in γ on the growth rates, keeping all other parameters same. A closer look at the table shows that as the weight of the environmental quality increases from 1.5 to 2.5, g_L, the low growth equilibrium increases; while g_H decreases. Hence decreasing the growth gap.

For two countries with a low σ (0.5), a developing economy can grow by giving emphasis to environmental quality (γ). The results show that as the individuals start to prefer better environmental quality and γ increases, the low growth rate gets better and the economy begins to move towards a higher and better growth rate (similar to that of a developed economy).

The figure 1.1 indicates a clear picture of the multiple equilibria through changes in γ.

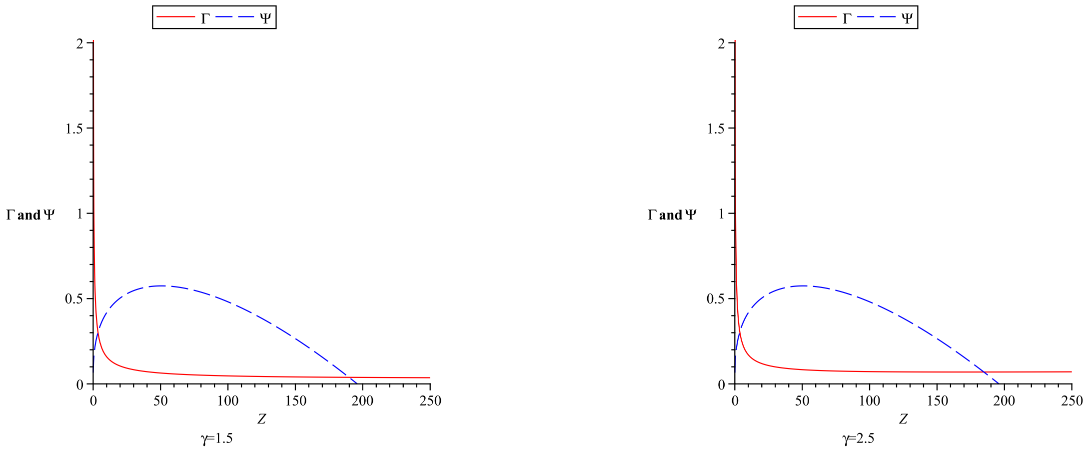

Figure 1.1: Effect of change in γ on multiple equilibria when $\alpha = 0.35, \beta = 0.1, \breve{\rho} = 0.001, \sigma = 0.5, \tau = 0.022, \theta = 0.017, \nu = 1$

Similarly, table 1.2 also observes changes in growth rates through changes in σ, while keeping all other parameters the same.

It is observed that as the intertemporal elasticity of substitution increases (or σ decreases), the distance between the equilibria shrinks. As σ decreases from 0.5 (the benchmark value) to 0.4, the low growth equilibrium increases from 0.78% to 0.79% and the high growth equilibrium decreases in value from 3.13% to 3.12%.

This indicates that as σ decreases, which increases the intertemporal elasticity of substitution, there is increased substitutability between consumption and saving. Hence individuals are willing to consume less and save more thus increasing the growth rate.

The figure (1.2) further illustrate the changes. The functions Γ and Ψ can be observed in the graph and the changes due to the increase in σ can clearly be seen.

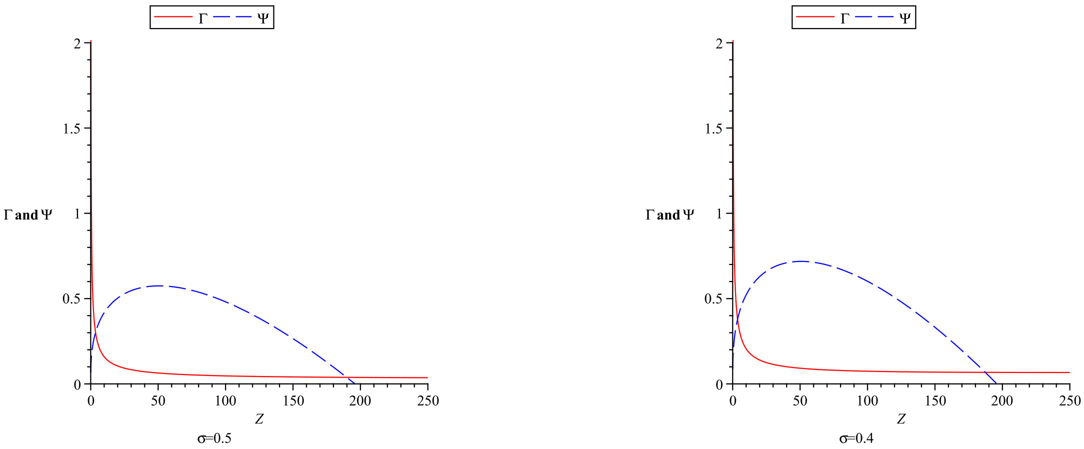

Figure 1.2: Effect of change in σ on multiple equilibria when $\alpha = 0.35, \beta = 0.1, \breve{\rho} = 0.001, \gamma = 1.5, \tau = 0.022, \theta = 0.017, \nu = 1$

In additions, the table 1.2 also measures how changes in β, the slope of the impatience function, affect the equilibrium.

The simulations will help evaluate changes in growth rates by changing the value of β, ceteris paribus.

In comparison to table 1.1, the table (1.2) observes that as β decreases from 0.15 to 0.1 (the benchmark parameter), the low growth equilibrium increases from 0.68% to 0.78%. On the other hand, the high growth equilibrium decreases from 3.15% to 3.14%. Thus the gap between both BGPs decreases as the individual becomes more patient.

As β decreases the individuals become more patient, possibly leading to a higher growth equilibrium by consuming less and thereby increasing saving propensity.

The figure (1.3) show how β effects the equilibria.

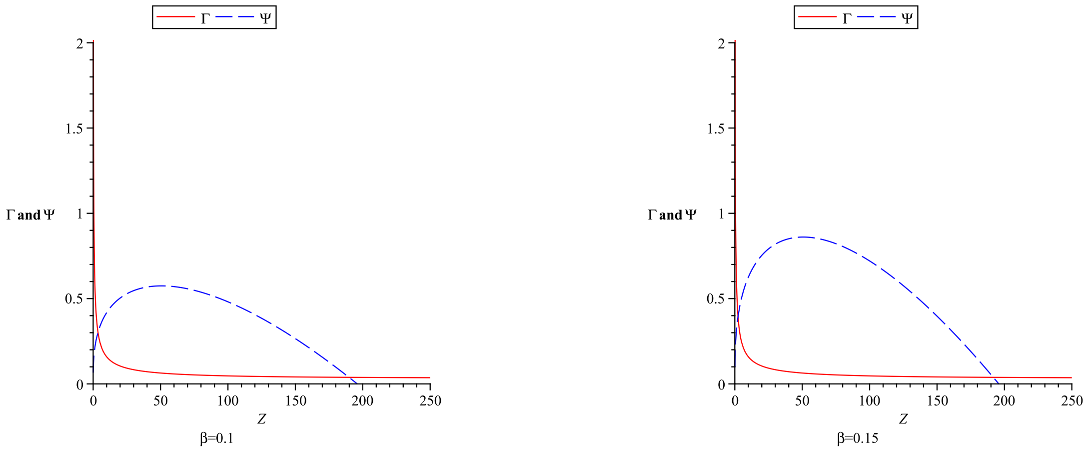

Figure 1.3: Effect of change in β on multiple equilibria when $\alpha = 0.35, \sigma = 0.5, \breve{\rho} = 0.001, \tau = 0.022, \theta = 0.017, \gamma = 1.5, \nu = 1$

In essence, changes in β show that an individual becomes more patient as the slope of the indifference curve decreases. This indicates that current consumption is less and the individual saves for the future. Also, change in β shows that when individuals are more patient, the growth rates tend to converge to a stable BGP as well.

1.5 Conclusion

This chapter has explained in detail the model and and its solution. In addition, simulations have been done to observe how changing the weights on several parameters (γ, σ, β) results in changes in the multiple growth rates of the model; and how high growth can be achieved by any economy.

In essence the results from this chapter show that a decrease in the time preference rate, increases the patience of the individual, increasing future savings and reducing current consumption. It also increases the low growth rate and shrinks the difference between both equilibria.

Furthermore, the same result is observed when the individual weighs the environment more (i.e. γ); and when the individual prefers to save more and consume less in the current time period (i.e. σ decreases).

The question now is to interpret which equilibria are stable out of the multiple BGPs seen. Also, how we can improve low quality equilibria, if that is stable; and reach the high growth equilibrium. Hence, a stability analysis is needed to answer such questions. The next chapter will look at and observe how and which of the equilibria are judged to be stable for the above stated simulations and results.

Chapter 2

Stability analysis and transitional dynamics

In this chapter, the stability and transitional dynamics around the equilibrium, are analyzed. By changing the weights of the parameters, namely the inverse of intertemporal elasticity of substitution,σ, the weight assigned to stock of environment, γ, and the slope of the impatience function, β.

2.1 Stability and transitional dynamics around equilibrium

This section begins to study the the local stability around the balanced growth path (BGP). We do this by using the reduced linearized equations (1.18) and (1.19) around the steady state values of X^* and Z^*.

The stability and transitional dynamics around equilibrium, can be said to be dependant on the Jacobian Matrix for this system (Chaudhry et al., 2017). This is as follows:

$$J\mid_{(X^*,Z^*)}=\begin{bmatrix} \frac{\partial \dot{X}}{\partial X}\mid_{(X^*,Z^*)} & \frac{\partial \dot{X}}{\partial Z}\mid_{(X^*,Z^*)} \\ \frac{\partial \dot{Z}}{\partial X}\mid_{(X^*,Z^*)} & \frac{\partial \dot{Z}}{\partial Z}\mid_{(X^*,Z^*)} \end{bmatrix}, \tag{2.1}$$

where the Jacobian Matrix for the model is given in equation 2.2 below:

$$J^* = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{2.2}$$

here,

$$a_{11} = \frac{\partial \dot{X}}{\partial X} = X^* + \frac{\rho'(.)X^*Z^*}{\nu(1-\sigma)-1}$$

$$\begin{gathered} a_{12} = \frac{\partial \dot{X}}{\partial Z} = \frac{1}{\nu(1-\sigma)-1}X^*[-\gamma(1-\sigma)(\tau-\theta)\alpha Z^{*\alpha-1} \\ -\alpha(\alpha-1)(1-\tau)Z^{*\alpha-2} + \rho'(.)X^*] - (1-\tau)(\alpha-1)Z^{*\alpha-2}X^* \end{gathered} \tag{2.3}$$

$$a_{21} = \frac{\partial \dot{Z}}{\partial X} = -Z^*$$

$$a_{22} = \frac{\partial \dot{Z}}{\partial Z} = (\alpha-1)X^* - (\tau-\theta)Z^{*\alpha}$$

This linearized system of equations include one control variable, X, and one state variable, Z. Hence, there are three possibilities for the local dynamics:

(1) if $det < 0$ than it will be locally saddle path stable,

(2) if $det > 0$ and $trace > 0$ than it will be locally unstable and,

(3) if $det > 0$ and $trace < 0$ than it will be locally indeterminate.

Calculating the determinant of 2×2 matrix, gives us:

$$\begin{aligned} detJ^* = X^*Z^*[-\alpha(\tau-\theta) + \frac{\rho'(.)}{\nu(1-\sigma)-1}(\alpha(1-\tau) \\ -(\alpha+1)(\tau-\theta)Z) + \frac{\alpha(1-\alpha)(1-\tau)}{1-\nu(1-\sigma)}Z^{\alpha-2} \\ +\frac{\alpha\gamma(1-\sigma)(\tau-\theta)}{1-\nu(1-\sigma)}] \end{aligned} \tag{2.4}$$

The trace of the Jacobian matrix, $tr(J^*)$ is given by:

$$tr(J^*) = \alpha(1-\tau)Z^{*\alpha-1} - (\alpha+1)(\tau-\theta)Z^{*\alpha} + \frac{X^*Z^*\rho'(.)}{\nu(1-\sigma)-1} \tag{2.5}$$

Also since the signs of $tr(J^*)$ and $det(J^*)$, are difficult to determine, the numerical simulations can be used here to access the stability of the BGPs.

2.2 Numerical simulations

In this section we analyze the unique and multiple equilibria cases that have been evaluated and discussed in Chapter 1. Similar to Chapter 1, these simulations will be undertaken through changes observed through the parameters: γ, σ, and β.

In addition, we will observe these changes on the same set of parameter values as in table (1.1). The subsequent sections provide results for numerical simulations.

Under the stability analysis, the values of the growth rates, the related values of X^* and Z^*, and the signs of $det(J^*)$ and $tr(J^*)$ are observed and interpreted.

2.2.1 Unique equilibria and stability analysis

This section observes the unique equilibrium (where $\gamma \leq 0.36$). For this section we observe the stability of the equilibrium through these values of the parameters: $\alpha = 0.35, \beta = 0.1, \breve{\rho} = 0.001, \gamma = 0.36, \sigma = 0.5, \tau = 0.022, \theta = 0.017, \nu = 1$.

Table 2.1: Equilibrium properties under unique equilibrium

$\alpha = 0.35, \beta = 0.1, \breve{\rho} = 0.001, \gamma = 1.5, \sigma = 0.5, \tau = 0.022, \theta = 0.017, \nu = 1$

	Low growth equilibrium
Growth rate(%)	0.77
X(C/K)	0.4269
Z(K/E)	3.4820
Consumption/GDP(C/Y)	0.9606
Private investment/GDP(K'/Y)	0.0174
Consumption/Environemntal Quality(C/E)	1.4865
$\hat{\rho}$	0.1497
$detJ^*$	-0.1273
$tr(J^*)$	-0.1556
Local property	Saddle path stable

As per table 2.1, there is only one unique equilibrium, the low growth equilibrium. This is saddle path stable, as $detJ^* < 0$ and $tr(J^*) < 0$.

Furthermore, the rate of time preferences is 0.15, for this low equilibrium, indicating that individuals are impatient. For these values of parameters, it is seen that the consumption to environmental quality (C/E) ratio is almost 1.5, indicating a marked preference for consuming in the current time period, and hence saving less.

2.2.2 Multiple equilibria and stability analysis

This section measures the stability analysis of changes in parameters under conditions of multiple equilibria. The benchmark values of table (1.1) are used to compare how stability of values is affected. The table 2.2 gives the equilibrium properties under multiple equilibrium at the benchmark values of the parameters.

Table 2.2: Equilibrium properties under multiple equilibrium

$\alpha = 0.35, \beta = 0.1, \breve{\rho} = 0.001, \gamma = 1.5, \sigma = 0.5, \tau = 0.022, \theta = 0.017, \nu = 1$

	Low growth equilibrium	High growth equilibrium
Growth rate(%)	0.78	3.14
X	0.4184	0.0010
Z	3.5882	189.8121
C/Y	0.9601	0.0289
K'/Y	0.0179	0.9491
C/E	1.5014	0.1815
$\hat{\rho}$	0.1511	0.0192
$detJ^*$	-0.1222	0.0011
$tr(J^*)$	-0.1617	-0.0673
Local property	Saddle path stable	Stable

In table 2.2, under conditions of multiple equilibria, we note that the consumption to capital (X^*) ratio is higher in the low growth equilibrium. This indicates more spending on consumption than capital. Similarly consumption to GDP ratio and the capital to environmental quality ratio (Z^*), are higher for the low growth rate. Also indicating that the individual in the low growth equilibrium invests more in the current time period.

This low growth equilibrium is still saddle path stable . However, the high growth rate at 3.14% is a stable node with $detJ > 0$, and the $tr(J^*) < 0$.

Also, the rate of time preferences is 0.15, for the low equilibrium, and 0.02 for the high equilibrium, indicating that the individuals are more patient in the stable high equilibrium. The individuals thus consume less (C/E ratio is

0.18) in the current time period and save for the future.

We now evaluate the stability of the observed results in Chapter 1 in the tables below. First we observe how changes in γ affect the equilibrium properties. The table 2.3, below, shows how changing γ from 1.5 to 2.5, affects certain elements of the model.

Table 2.3: Effect of change in γ

$\alpha = 0.35, \beta = 0.1, \breve{\rho} = 0.001, \gamma = 2.5, \sigma = 0.5, \tau = 0.022, \theta = 0.017, \nu = 1$

	Low growth equilibrium	High growth equilibrium
Growth rate(%)	0.79	3.11
X	0.4109	0.0018
Z	3.6870	184.7157
C/Y	0.9596	0.0544
K'/Y	0.0184	0.9236
C/E	1.5150	0.3381
$\hat{\rho}$	0.1525	0.0348
$detJ^*$	-0.1178	0.0021
$tr(J^*)$	-0.1671	-0.0980
Local property	Saddle path stable	Stable

The table 2.3 follows a similar trend in variables as the previous trend seen in table 2.2.

A comparison between table 2.2 and table 2.3 shows, that as γ increases from 1.5 to 2.5, the growth gap shrinks. Essentially, the low growth equilibria increases from 0.78% to 0.79%; while the high growth equilibria decreases from 3.14% to 3.11%.

The table 2.4 observes the changes in the equilibrium properties and the stability of the equilibrium because of changes in σ, keeping all other parameters same.

Table 2.4: Effect of change in σ

$\alpha = 0.35, \beta = 0.1, \breve{\rho} = 0.001, \gamma = 1.5, \sigma = 0.5, \tau = 0.022, \theta = 0.017, \nu = 1$

	Low growth equilibrium	High growth equilibrium
Growth rate(%)	0.79	3.12
X	0.4147	0.0014
Z	3.6370	187.2646
C/Y	0.9598	0.0417
K'/Y	0.0182	0.9363
C/E	1.5082	0.2602
$\hat{\rho}$	0.1518	0.0270
$detJ^*$	-0.1500	0.0020
$tr(J^*)$	-0.2400	-0.0958
Local property	Saddle path stable	Stable

As σ decreases from 0.5 to 0.4 (a comparison of tables 2.2 and 2.4, respectively) it is observed that the growth gap decreases, as low growth goes up from 0.78% to 0.79%, and the high growth goes down from 3.13% to 3.12%.

Hence, a change in σ (decrease from 0.5 to 0.4) shows that individuals are choosing to smooth consumption over their lifetime, and hence saving more.

Table 2.5: Effect of change in β

$\alpha = 0.35, \beta = 0.15, \breve{\rho} = 0.001, \gamma = 1.5, \sigma = 0.5, \tau = 0.022, \theta = 0.017, \nu = 1$

	Low growth equilibrium	High growth equilibrium
Growth rate (%)	0.68	3.15
X	0.5513	0.0006
Z	2.3703	191.7652
C/Y	0.9661	0.0192
K'/Y	0.0119	0.9588
C/E	1.3068	0.1207
$\hat{\rho}$	0.1970	0.0191
$detJ^*$	-0.2124	0.0011
$tr(J^*)$	-0.2058	-0.0675
Local property	Sadle path stable	Stable

From changes in β under multiple equilibria, it is observed that as β decreases from 0.15 to 0.1 (see tables 2.5 2.2, respectively), the gap between both equilibria decreases. Furthermore, the low growth equilibria increases from 0.68% to 0.78%; and the high growth equilibria increases from 3.15% to 3.14%.

In addition, as β decreases from 0.15 to 0.1, the rate of time preferences decreases for both BGPs, making the individual with $\beta = 0.1$ more patient.

Thus as β decreases, the individual becomes more patient and prefers to consume less in the current time period, hence saving more for the future; and hence resulting in a more stable high growth equilibrium.

These multiple BGPs, and changes in growth due to weight given to environmental quality, the intertemporal elasticity of substitution, and the slope of the impatience function, can explain the growth in multiple developed or developing countries.

2.3 Conclusion

The results of this model are informative as they can be seen to explain economic growth of countries which range from a low growth and high consumption equilibrium, to countries that have high growth and low consumption equilibrium.

The difference in these economies is the weight they have assigned to the environmental quality (γ), the intertemporal elasticity of substitution (σ), and their time preferences (β) in their production functions.

For the low growth case, the weight assigned to the environmental quality, γ, is low ($\gamma = 1.5$). Furthermore, the high growth equilibrium, will be reached as individuals start to prefer a better environmental quality, i.e. as γ increases ($\gamma = 2.5$).

The results also indicate that as the inverse of intertemporal elasticity of substitution decreases (σ), or the intertemporal elasticity of substitution increases, there is a consumption smoothing effect and the individual prefers to consume less and save more, leading to a higher and stable equilibrium.

Also, as the slope of the impatience function decreases, the individual becomes more patient and prefer to consume less in the current time period, leading to a higher growth rate.

The results essentially show how changing certain parameters, affect the growth rates of an economy, leading to high growth low consumption equilibria (or a better BGP).

This chapter shows which equilibria are stable out of the two multiple equilibria from chapter 1's solution. The results from changing parameters show that mostly a high growth equilibrium is stable. However, there is still a possibility for the government to step in. Fiscal policies are known to play a role in maximizing growth, under certain equilibrium conditions.

Chapter 3

Fiscal implications

This chapter takes up from the results of the previous chapter and elaborates on a role that makes the government important. Chaudhry et al. (2017) also mention how the government can play a role in economic growth especially, in models that incorporate economic growth. Similarly Dioikitopoulos and Kalyvitis (2010) also indicate that their findings can have fiscal implications.

The model solved in the chapters 1 and 2, shows that unique and multiple equilibria both exist for this system. This means that for multiple equilibria, environment's effect on the growth rate and subsequently the time preferences of the individual can result in the possibility of both a high and low environmental quality, and a high and low growth equilibria to exist simultaneously. Hence the government can play a role here, and government intervention can possibly lead to a higher growth equilibrium or a decrease in the distance between low and high growth equilibria (Dioikitopoulos and Kalyvitis, 2015; Vella et al., 2015).

Fiscal implications here can include a growth maximizing fiscal policy in lieu of the environmental policy and individual preferences (Dioikitopoulos et al., 2016).

This chapter will look at a growth enhancing fiscal policy by considering what Dioikitopoulos and Kalyvitis (2015) has done. Dioikitopoulos and Kalyvitis (2015) defines "A growth maximizing (GM) allocation when (i) the government chooses the tax rate and aggregate allocations in order to maximize the growth rate of the economy by taking into account the aggregate optimality conditions of the CDE, and (ii) the government budget constraint and the feasibility and technological conditions are met."

The sections ahead solve for the growth maximizing fiscal policy, and see which work best in presence of time preferences and environmental quality. I will then further review the optimal fiscal policy through numerical simulations.

3.1 Growth maximizing fiscal policy

In this section we solve for the growth maximizing fiscal policy. For this we solve a static optimization problem subject to a single constraint.

The problem is as follows:

$$Max \quad g = (\tau - \theta)Z^{\alpha} \tag{3.1}$$

Subject to

$$\begin{aligned}\frac{\alpha}{1-\nu(1-\sigma)}(1-\tau)Z^{\alpha-1} + \frac{\gamma}{1-\nu(1-\sigma)}(1-\sigma)(\tau-\theta)Z^{\alpha} \\ -\frac{\breve{\rho}}{1-\nu(1-\sigma)} - \frac{\beta(1-\tau)Z^{\alpha}}{1-\nu(1-\sigma)} - \frac{\beta(\tau-\theta)Z^{\alpha+1}}{1-\nu(1-\sigma)} - (\tau-\theta)Z^{\alpha} = 0\end{aligned} \tag{3.2}$$

So the static optimization Lagrange comes out to be:

$$\begin{aligned}L = (\tau-\theta)Z^{\alpha} + \lambda[\frac{\alpha}{1-\nu(1-\sigma)}(1-\tau)Z^{\alpha-1} \\ +\frac{\gamma}{1-\nu(1-\sigma)}(1-\sigma)(\tau-\theta)Z^{\alpha} - \frac{\breve{\rho}}{1-\nu(1-\sigma)} - \frac{\beta(1-\tau)Z^{\alpha}}{1-\nu(1-\sigma)} \\ -\frac{\beta(\tau-\theta)Z^{\alpha+1}}{1-\nu(1-\sigma)} - (\tau-\theta)Z^{\alpha}]\end{aligned} \tag{3.3}$$

The first order conditions (FOCs) will result in an equation of λ given as equation 3.4 (See Appendix C for further details):

$$\lambda = \frac{Z(1-\nu(1-\sigma))}{\alpha + Z - \nu Z + \nu\sigma Z - \beta Z - \beta Z^2 - \gamma Z + \gamma\sigma Z} \tag{3.4}$$

And finally the growth optimizing τ comes out to be:

$$\tau = \frac{\alpha\beta Z + \theta\beta Z^2 + \alpha^2\theta - \alpha^2 + \alpha - \alpha\theta\beta Z}{\alpha + \beta Z^2} \tag{3.5}$$

This is the growth maximizing tax rate that the government can pursue (with respect to all the parameters) as an optimal policy, in order to achieve optimal growth rate.

In order to look at τ with respect to the unique and multiple growth, we once again look at the simulations.

3.2 Numerical simulations for fiscal implications

This section uses simulations to explain the role of the government's growth maximizing fiscal policy on unique and multiple equilibria. In particular we can examine the changes in the slope of the impatience function, the intertemporal elasticity of substitution, and the weight given to the environmental quality.

As in the previous sections, we use the parameter values as given in table (1.1). On the basis of these parameters, we change the value of σ, and β, and observe changes in the optimal tax rate and eventually the growth rate.

3.2.1 Growth Maximizing Fiscal Policy under Multiple Equilibria

This section will look at the changes in the Growth Maximizing (GM) fiscal policy because of changes in σ, and β.

Dioikitopoulos and Kalyvitis (2015) also examine the effects of government tax policy under endogenous time preferences. Similarly, this section observes how government policy (fiscal) can be used to obtain optimal growth in the economy.

The table 3.1, below, shows changes in the growth maximizing fiscal policy, or $\hat{\tau}$.

Table 3.1: Effect of change in parameters

	g	$\hat{\tau}$	Z^*	X^*	C/E	$\hat{\rho}$
Benchmark Values $\sigma = 0.5, \beta = 0.1$	0.1101	0.0643	11.1900	0.0846	0.9468	0.0957
$\sigma = 0.4, \beta = 0.1$	0.0947	0.0555	13.1156	0.0826	1.0837	0.1094
$\sigma = 0.5, \beta = 0.15$	0.1398	0.0859	7.5522	0.1058	0.7994	0.1209

The decrease in σ, the inverse of intertemporal elasticity of substitution, follows the same pattern as described in Dioikitopoulos and Kalyvitis (2015). It is observed that as the intertemporal elasticity of substitution increases, the individuals prefer to consume less in the current time period. In this case, the government sets a lower growth maximizing tax rate and the economy converges to a lower stable steady state equilibrium along the BGP.

A decrease in β, the slope of the impatience function, decreases the rate of time preference, making the individual more patient. In the case under analysis, as β increases, the individual becomes impatient, the government steps in by increasing the tax rate and thus the growth rate.

Summarizing the results in table 3.2, gives a clearer picture of the direction of movement of growth maximizing tax rate as the parameters σ and β are changed through numerical simulations.

Table 3.2: Summary of change in parameters' GM allocation responses

	g^{GM}	$\hat{\tau}$	Z^*	X^*	C/E	$\hat{\rho}$
Increase in σ	$(+)$	$(+)$	$(-)$	$(+)$	$(-)$	$(-)$
Increase in β	$(+)$	$(+)$	$(-)$	$(+)$	$(-)$	$(+)$

It is important to note the changes due to β. As the slope of the impatience function increases, it makes the individual more impatient, thus government will increase its taxes to increase growth rates.

3.3 Conclusion

In this section, I observe a growth maximizing fiscal policy for this model. The tax rate is then seen as the parameters are varied. Here the optimal tax rate and the resulting growth rates are observed when σ, the inverse of intertemporal elasticity of substitution, and β, slope of the impatience function, are changed.

CONCLUSIONS

Environmental quality affects economic growth through various channels. Here, I look at a model for economic growth, effected by environmental quality while taking into account time preferences of individuals. I also measures changes in this relationship if fiscal implications are present.

I extend the model of Chaudhry et al. (2017) by taking into account the time preferences. Furthermore, numerical simulations are undertaken to explore the theoretical basis of the model and its solutions. The simulations can help provide a clearer picture of the unique or multiple BGPs. The model results in both a unique and multiple equilibria.

Essentially, two economies can exist for certain parameters, one with low growth high consumption, and the other with low consumption and high growth. These growth rates can then be manipulated through changes in the environmental quality, the inverse of intertemporal elasticity of substitution, and the slope of the time preference function.

Finally, this model also finds a growth maximizing fiscal policy. The existence of multiple equilibria suggests that the government can play a role in improving the growth rate of the economy, its tool being the fiscal policy. The growth maximizing fiscal policy is also analyzed with changes in the inverse of intertemporal elasticity of substitution, and the slope of the impatience (time preference) function.

The results indicate that individuals are more patient as the environmental quality gets better, and hence the economy moves to the better equilibrium along the balanced growth path. It is seen that depending on preferences, a low growth and high consumption equilibrium exists when individual gives less wight to environmental quality. Thus as preference for a cleaner environment grows, one can see the economy move to better quality equilibrium with low consumption in current time period and more saving for the future.

In addition, under certain conditions reflecting an individuals time preferences, a growth maximizing fiscal policy can be effective in reaching a better equilibrium.

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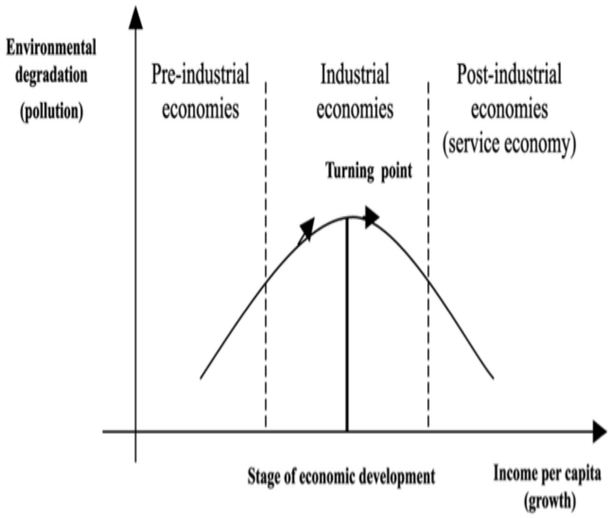
Appendix A

The Environmental Kuznets Curve is given below.



Source: Panayotou (1993)

Appendix B

Hence for the model the necessary first order conditions for optimal control are:

$$\frac{\partial H}{\partial C} = \nu E^{\gamma(1-\sigma)} C^{\nu(1-\sigma)-1} e^{-\Delta(t)} - \lambda = 0 \tag{B-1}$$

The above equation shows that the shadow price comes out to be:

$$\lambda = \nu E^{\gamma(1-\sigma)} C^{\nu(1-\sigma)-1} e^{-\Delta(t)} \tag{B-2}$$

$$\dot{K} = (1-\tau)K^{\alpha}E^{1-\alpha} - C, \ , 0 < \tau < 1,$$

$$\dot{\lambda} = -\lambda\alpha(1-\tau)K^{\alpha-1}E^{1-\alpha} \tag{B-3}$$

Solving the model further and differentiating equation with respect to time t, the growth rate of consumption is

$$\frac{\dot{C}}{C} = \frac{1}{\nu(1-\sigma)-1}(\frac{\dot{\lambda}}{\lambda} - \gamma(1-\sigma)\frac{\dot{E}}{E} - \rho(\frac{C}{E})) \tag{B-4}$$

Hence as per the equation above, the growth rate of consumption is dependent upon growth rate of stock of environment, the growth rate of shadow price of the physical capital, and the weight assigned to stock of environment, which is γ, and also to the inverse of intertemporal elasticity of substitution, which is σ.

Now substituting the values of $\frac{\dot{E}}{E}$ and $\frac{\dot{\lambda}}{\lambda}$ in the equation above, the growth rate of consumption comes out to be:

$$\frac{\dot{C}}{C} = \frac{1}{\nu(1-\sigma)-1}[-\gamma(1-\sigma)(\tau-\theta)(\frac{K}{E})^{\alpha} - (\alpha)(\frac{K}{E})^{\alpha-1}(1-\tau) + \rho(\frac{C}{E})] \quad \text{(B-5)}$$

Thus the system of equations for the model are as follows:

$$\frac{\dot{K}}{K} = (1-\tau)\frac{K}{E}^{\alpha-1} - \frac{C}{K}, \quad \text{(B-6)}$$

$$\frac{\dot{C}}{C} = \frac{1}{\nu(1-\sigma)-1}[-\gamma(1-\sigma)(\tau-\theta)(\frac{K}{E})^{\alpha} - (\alpha)(\frac{K}{E})^{\alpha-1}(1-\tau) + \rho(\frac{C}{E})], \quad \text{(B-7)}$$

$$\frac{\dot{E}}{E} = (\tau-\theta)(\frac{K}{E})^{\alpha}. \quad \text{(B-8)}$$

As a solution, these yields and reduces to a system of two equations:

$$\frac{\dot{X}}{X} = \frac{1}{\nu(1-\sigma)-1}[-\gamma(1-\sigma)(\tau-\theta)Z^{\alpha} - \alpha(1-\tau)Z^{\alpha-1} + \rho(XZ)] - (1-\tau)Z^{\alpha-1} + X \quad \text{(B-9)}$$

$$\frac{\dot{Z}}{Z} = (1-\tau)Z^{\alpha-1} - X - (\tau-\theta)Z^{\alpha} \quad \text{(B-10)}$$

We have two functions that are satisfied by the value of Z, namely $\Gamma(Z)$ and $\Psi(Z)$. These are given by:

$\Gamma(Z) = \frac{\alpha(1-\tau)Z^{\alpha-1}}{1-\nu(1-\sigma)} - (\tau-\theta)Z^{\alpha} + \frac{\gamma(1-\sigma)(\tau-\theta)Z^{\alpha}}{1-\nu(1-\sigma)}$,

$\Psi(Z) = \frac{\rho(t)}{1-\nu(1-\sigma)}$

Here now $\rho(t) = \beta(XZ) + \tilde{\rho}$.

These functions are both continuous in Z, and since for this to be true the function $X(Z)$ needs to be greater than zero, hence

Since

$$X = (1-\tau)Z^{\alpha-1} - (\tau-\theta)Z^{\alpha} \tag{B-11}$$

Then

$$Z < \frac{1-\tau}{\tau-\theta} \tag{B-12}$$

Therefore:

$\lim_{Z\to 0}\Gamma(Z) = 0$, $\lim_{Z\to\frac{1-\tau}{\tau-\theta}}\Gamma(Z) = (\frac{\alpha(1-\tau)^{\alpha}}{1-\nu(1-\sigma)(\tau-\theta)^{\alpha-1}}) + \frac{\gamma(1-\sigma)(\tau-\theta)^{1-\alpha}(1-\tau)^{\alpha}}{1-\nu(1-\sigma)}$

and:

$\frac{\partial\Gamma(Z)}{\partial Z} < 0, \frac{\partial^2\Gamma(Z)}{\partial Z^2}$ Also for $\Psi(Z)$

1. $\lim_{Z\to 0}\Psi(Z) = \frac{\tilde{\rho}}{1-\nu(1-\sigma)}$, $\lim_{Z\to\frac{1-\tau}{\tau-\theta}}\Psi(Z) = \frac{1}{1-\nu(1-\sigma)}[\frac{\beta(1-\tau)^{\alpha+1}}{(\tau-\theta)^{\alpha-2}} - \frac{\beta(1-\tau)^{\alpha}}{(\tau-\theta)^{\alpha-1}} + \tilde{\rho}]$

2. Since $\frac{\partial\Psi(Z)}{\partial Z} = \frac{\beta[\alpha(1-\tau)Z^{\alpha-1}+(\alpha+1)(\tau-\theta)Z^{\alpha}]}{1-nu(1-\sigma)}$. In essence, for $\alpha(1-\tau)Z^{\alpha-1} + (\alpha+1)(\tau-\theta)Z^{\alpha} > 0$, $Z < \frac{\alpha(1-\tau)}{(\alpha+1)(\tau-\theta)}$.

Appendix C

Solving for the growth maximizing fiscal policy. For this we solve a static optimization problem subject to a single constraint.

The problem is as follows:

$$Max\ g = (\tau - \theta)Z^{\alpha}, \tag{C-1}$$

Subject to

$$\frac{\alpha}{1-\nu(1-\sigma)}(1-\tau)Z^{\alpha-1} + \frac{\gamma}{1-\nu(1-\sigma)}(1-\sigma)(\tau-\theta)Z^{\alpha} - \frac{\hat{\rho}}{1-\nu(1-\sigma)} - \frac{\beta(1-\tau)Z^{\alpha}}{1-\nu(1-\sigma)} - \frac{\beta(\tau-\theta)Z^{\alpha+1}}{1-\nu(1-\sigma)} - (\tau-\theta)Z^{\alpha} = 0. \tag{C-2}$$

So the static optimization Lagrange comes out to be:

$$L = (\tau-\theta)Z^{\alpha} + \lambda[\frac{\alpha}{1-\nu(1-\sigma)}(1-\tau)Z^{\alpha-1} + \frac{\gamma}{1-\nu(1-\sigma)}(1-\sigma)(\tau-\theta)Z^{\alpha} - \frac{\hat{\rho}}{1-\nu(1-\sigma)} - \frac{\beta(1-\tau)Z^{\alpha}}{1-\nu(1-\sigma)} + \frac{\beta(\tau-\theta)Z^{\alpha+1}}{1-\nu(1-\sigma)} - (\tau-\theta)Z^{\alpha}]. \tag{C-3}$$

The First Order Conditions (FOCs) are as follows:

$$\frac{\partial L}{\partial Z} = \alpha(\tau-\theta)Z^{\alpha-1} + \lambda[\frac{\alpha(\alpha-1)}{1-\nu(1-\sigma)}(1-\tau)Z^{\alpha-2} + \frac{\alpha\gamma}{1-\nu(1-\sigma)}(1-\sigma)(\tau-\theta)Z^{\alpha-1} - \frac{\alpha\beta(1-\tau)Z^{\alpha-1}}{1-\nu(1-\sigma)} - \frac{(\alpha+1)\beta(\tau-\theta)Z^{\alpha}}{1-\nu(1-\sigma)} - \alpha(\tau-\theta)Z^{\alpha-1}] = 0, \tag{C-4}$$

$$\frac{\partial L}{\partial \tau} = Z^{\alpha} + \lambda[\frac{-\alpha Z^{\alpha-1}}{1-\nu(1-\sigma)} + \frac{\gamma}{1-\nu(1-\sigma)}(1-\sigma)Z^{\alpha} + \frac{\beta Z^{\alpha}}{1-\nu(1-\sigma)} + \frac{\beta Z^{\alpha+1}}{1-\nu(1-\sigma)} - Z^{\alpha}] = 0, \tag{C-5}$$

$$\frac{\partial L}{\partial \lambda} = \frac{\alpha}{1-\nu(1-\sigma)}(1-\tau)Z^{\alpha-1} + \frac{\gamma}{1-\nu(1-\sigma)}(1-\sigma)(\tau-\theta)Z^{\alpha} - \frac{\hat{\rho}}{1-\nu(1-\sigma)} - \frac{\beta(1-\tau)Z^{\alpha}}{1-\nu(1-\sigma)} - \frac{\beta(\tau-\theta)Z^{\alpha+1}}{1-\nu(1-\sigma)} - (\tau-\theta)Z^{\alpha} = 0 \tag{C-6}$$

From these FOCs, we get λ as:

$$\lambda = \frac{Z(1-\nu(1-\sigma))}{\alpha + Z - \nu Z + \nu\sigma Z - \beta Z - \beta Z^2 - \gamma Z + \gamma\sigma Z}. \tag{C-7}$$

And finally the growth optimizing τ comes out to be:

$$\tau = \frac{\alpha\beta Z + \theta\beta Z^2 + \alpha^2\theta - \alpha^2 + \alpha - \alpha\theta\beta Z}{\alpha + \beta Z^2}. \tag{C-8}$$