Multiple equilibria and stability analysis of an economic growth model with Human capital externalities and endogenous time preferences

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DECLARATION

I undertake that this thesis is my own work. This has not been written by any other person. I also undertake that all the content taken from published and unpublished sources has been properly acknowledged. It is being submitted to Lahore School of Economics for the completion for MPhil Economics. And this work has not been submitted for any degree at any other institute.

Ammara Riaz

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C(t): Consumption Level

H(t): Stock of Human capital

Y(t): Output in the economy

K(t): Level of Physical capital

m: Discount factor

σ: Inverse of intertemporal elasticity of substitution

α: Weight assigned to physical capital

γ: Weight assigned to the stock of Human capitl

ϵ: Degree of externality

ϕ: Efficiency parameter

Abstract

This thesis develops a model to analyze the growth and fiscal policy implications in a model in which there exists an externality through the aggregate human capital. We endogenize the individual rate of time preferences and our results show that individuals that are more patient invest more in human capital. The model is solved along the balanced growth path. There exist a low growth equilibrium in which agents put less weight on human capital in the utility function. Whereas there exist multiple growth paths if agents value human capital more. Further, we perform a stability analysis. The results of the model find that human capital generates positive externalities. Our analysis implies that in the presence of these externalities there is lower optimal government taxation.

Introduction

Economics has been a study of looking at how growth is affected; ranging from the study of micro markets and economy to a study of how growth is affected at the macro level (Barro, 1990; Lucas, 1988). However, reasons for economic growth abound in literature, but in the late 1980's, the shift of how economic growth is affected, happened; since then the macroeconomists have been looking for the long run determinants of economic growth, namely physical capital initially and later human capital. Lately, human capital is considered as one of the most important factors in the development of economies; how the increase in human capital can lead to growth of an economy, more than the accumulation of physical capital. Human capital is in fact, denoted by Gary Becker as:

"Human capital is a measure of the economic value of an employee's skill set. This measure builds on the basic production input of labor measure where all labor is thought to be equal. The concept of human capital recognizes that not all labor is equal and that the quality of employees can be improved by investing in them; the education, experience and abilities of employees have economic value for employers and for the economy as a whole."

From the definition by Gary Becker above it seems that the accumulation of human capital can be a factor for providing a stable environment that can lead to the growth in the economy. Hence, in the labor market, educational attainment is one of the most significant credential for future job market success and earnings of the individual. These higher market returns encourage individuals to accumulate human capital today.

Lucas (1988), a pioneer in the thought behind human capital as the driver of growth, considered that human capital accumulation directly affects productivity, which in turn affects the economy's growth. Similarly, Romer (1989), who took up the Lucas model, used a theoretical framework to show how endogenizing human capital can lead to growth in the economy. The initial level of literacy also plays a role in Romer 1989 model. Human capital accumulation leads directly to increased productivity Dinda (2008); however, human capital is also found to indirectly affect growth through spillover effects [Nelson and Phelps (1966); Becker and Mulligan (1997)]. Therefore the level of education, or an attainment and accumulation of human capital in an economy, can be used to explain the growth and income differences; these are studies in a seminal work by (Galor and Zeira, 1993).

Investment in human capital through more educational attainment and training could have a long-lasting and permanent impact. Workers could have a permanent effect on growth facilitated by intensive technological progress or either the adoption of new technologies by the skilled worker.

Initially in the classical theory of economic growth, labor productivity was given and exogenously determined; growth was affected by the ratio of the workforce and physical capital. The literature also lists many factors of growth (Barro, 1990; Lucas, 1988). However, education as a factor affecting growth, was not taken into account. More recently the neoclassical and the modern growth economists have highlighted the importance of education and innovation as the accumulation of human capital, which can affect long-term economic growth in an economy.

This relationship of human capital and economic growth has also been empirically studied in the literature. In the past decades, human capital growth in the organization for economic cooperation and development (OECD) countries was one of the key factors of development. The improvements in human capital accounted for almost half a percentage point acceleration in growth particularly in the 1980's. This pattern was found especially in Germany, Italy, Greece, Netherland, and Spain. Therefore, changes in human capital accumulation, macroeconomic conditions present in the economy, and the policies in effect, are important for different growth patterns in countries; and can hence explain a continuum of economic growth across countries. That in turn improves the living standards (Bassanini and Scarpetta, 2002).

Barro (2001) analyzed panel data of almost 100 countries for years 1965 to 1995. Education was taken as an investment in human capital that was measured by years of schooling. For a given level of GDP, a higher ratio of human capital to physical capital would result in higher growth. According to Barro This growth could be explained through two different channels. A high level of human capital may result in more absorption of better technology from other leading economies; this channel is where there is a leaning from developed economies of the technology already in use for better growth. The other possible channel is that economies with a higher level of human to physical capital may adjust the quantity of physical capital rapidly (Barro, 2001).

Fan et. al. (2016) empirically observed the relationship between quality of life, human capital and economic growth across the US; by creating an index for quality of life by calculating the amenity values. The results of the study confirmed the positive impact of human capital and quality of life on economic growth. The findings of the study had important policy implications; the better educated individuals (high human capital) support that developmental policy which was aimed at improving the quality of life (one such policy would be promoting a better quality of the environment).

The literature also used a panel data approach to investigate the role of human capital on economic growth (see Pelinescu (2015)). The results of the analysis confirmed a positive relationship between GDP per capita and human capital Pelinescu (2015).

The literature above proceeds to explain that human capital can have a significant effect on the economy's growth. Recent literature has taken this a step further by linking time preferences with growth. Fisher (1930) and Bohm-Bawerk (1971) defined time preferences as the marginal rate of substitution between current and future consumption.

Theories on an individual's preferences to save and invest in current or future time periods, are getting more attention in determining the theories of saving, investments, economic growth and many other related issues. The literature observes that Samuelson's (1937) work was a leading indication towards such an affect; the rates of time preferences were taken as given here. Later Uzawa (1968); Lucas and Stokey (1984) looked at the correlation between time preferences and consumption levels.

The theoretical literature highlights the importance of endogenous time preferences and long run growth. Dioikitopoulos and Kalyvitis (2015) analyzed the implications of endogenous time preferences for aggregate human capital; analyzing the impact of fiscal policy. The findings of Dioikitopoulos and Kalyvitis (2015) confirmed the existence of multiple balance growth paths in the presence of endogenous time preferences; also calculating a growth maximization tax rate feasible for growth along the BGP.

The impact of public policy on an individual's level of patience can also be viewed through the expenditure on health. Agénor (2010) confirmed that healthier individuals would value the future; this was linked with the individual being more patient as they expect higher life expectancy. Blanchard (1985) similarly emphasized the above relationship through an overlapping generations model (OLG) model using endogenous time preferences.

Many theoretical contributions have also been made in the literature to show the impact of human capital on the impatience level. This may be seen through the indirect channels of income and wealth. The following are the major theoretical contributions based on this assumption:Schumacher (2009); Strulik (2012); Hausman (1979) have shown that discount rates are inversely related to income level. Similarly,Horowitz (1991) and Pender (1996) found declining discount rates with wealth.

Human capital not only affects productivity and growth but has significant indirect effects through knowledge spillovers and externalities. As discussed by Audretsch et. al. (2012), the investment in human capital through more educational attainment and training have a long-lasting and permanent impact. So a more skilled worker can have a permanent effect on growth facilitated by intensive technological progress or through the absorption of new international knowledge by the skilled worker.

Kuo & Yang (2008) found that the absorptive ability of the human capital depends on acquiring advanced foreign technologies. The analysis shows the existence of R&D spillovers as well as international knowledge spillovers.

Apart from theoretical models, the existing empirical literature also provides strong evidence that education affects the patience level of the individuals. Fuchs (1982) first studied the relationship of time preferences and schooling empirically; the analysis confirmed that increased schooling makes individuals patient. Lawrance (1991) concluded that nonwhite families with no college education have an almost seven percentage point higher time preferences rate as compared to white families. **Harrison et. al. (2002) also confirmed the relationship between patience and education in Danish families. They find that highly educated adults are more patient than the less educated adults**. In recent empirical literature Meier & Sprenger (2010) also confirmed this relationship.Perez-Arce (2011) similarly examined whether the level of education of an individual makes him more patient or not. His analysis was based on a lottery in Mexico. His findings suggested that college education has a significant impact on time preferences.

There is also increasing evidence that the fiscal policy also has an affect on growth; through possibly the structure of taxation as well as the expenditure. Kneller et. al. (1999) found that growth increases when productive expenditure (including spending on education) is financed by non-distortionary taxes.

This thesis will use an endogenous growth model to study **human** capital with externalities and endogenous time preferences and we will also analyze the long run dynamics along the balanced growth path. In the steady state as discussed by Gaspar et. al. (2014)) all the variables would grow at the same rate. In our model, there would be multiple equilibria: high growth and low growth equilibria; indicating two economies with the same endowments of capital but in actuality experiencing different growth patterns due to differences in preferences of the individuals. Similarly, an analysis by Hosoya (2012) also shows the local dynamics of a growth model where the economy converges to high or low growth equilibria.

This thesis takes a step forward and adds to the existing literature in multiple ways. Firstly, a theoretical model is developed to see the impact **of** human capital and its externalities. Several literature strands analyze the impact of human capital externalities on the economy. Secondly, our model takes into account how patience could possibly impact the aggregate level of human capital in the economy. A patient individual would invest more in human capital, while an impatient person would do the opposite. Lastly, fiscal policy implications are drawn from the analysis to see how the growth of the economy could possibly maximize from these externalities.

This thesis is organized as **following**: In chapter 1 an economic growth model with human capital externalities is developed by also taking into account the endogenous time preferences. Further the dynamic properties of the model are seen through a numerical simulations analysis. These numerical simulations show how the growth gap is be affected by changing the parameters involved in the model. Chapter 2 looks at the stability analysis and numerical simulations to explain the impact of changing parameters impact on other variables. Lastly, chapter 3 is based on a government static optimization problem to maximize the growth rate in an economy; further explored through changes in certain parameters undertaken by once again the numerical simulations.

Chapter 1

Economic growth model with externalities and endogenous time preferences

This chapter sets up an economic growth model with physical and human capital stock. Moreover, this model gives special attention to human capital externalities that increase the accumulation of human capital in our economy with endogenous time preferences. The model will be used to study the equilibria and dynamics of the system.

1.1 Framework of the model

Human capital plays a vital role in the economic growth of countries. The differences between the growth rate of the two countries could be explained by the difference in the level of human capital accumulation along with time preferences. The model presented in this thesis proposes that human capital

generate externalities that in turn increases the accumulation of human capital and productivity in the economy.
In order to maximize the utility, we built a present value Hamiltonian which is maximized subject to the physical capital constraint along with endogenous time preferences. The time preferences are dependent upon consumption and human capital.

1.1.1 Production function

The economy is characterized by a standard Cobb-Douglas production function in which we are using two inputs, physical capital, and human capital, for the production of output:

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha} H(t)^{1-\alpha}, 0 < \alpha < 1, \tag{1.1}$$

where $Y(t)$ represents the aggregate output in the economy, $K(t)$ is the amount of physical capital, whereas the total labor force in the economy is given by $L(t)$ and $H(t)$ represents human capital stock. We are assuming that the population growth in our model is zero hence the labor is normalized to one, i.e. L(t)=1. Therefore all other variables in the production function are taken in per capita terms.

In our model $H(t)$ is not a choice variable in the optimization problem since individual agents in the economy take the overall stock of the human capital as given in their production function. So in the context of our model, by taking H exogenous, we mean that we are not taking first order conditions w.r.t $H(t)$. In the production function α is the weight assigned to the physical capital (which is also known as the output elasticity of production with respect to physical capital) and $1-\alpha$ is the weight assigned to human capital. Also, α lies between 0 and 1

1.1.2 Utility function

The utility function of a representative agent in the economy is :

$$U(C,H)=\frac{(C^{\nu}H^{\gamma})^{1-\sigma}-1}{1-\sigma}, \tag{1.2}$$

where C is consumption and H is human capital stock, ν is the weight given to consumption[1] and γ is weight assigned to human capital stock, H, in the utility function and σ is the inverse of the intertemporal elasticity of substitution.

1.1.3 Physical capital

The law of motion for the stock of physical capital can be written as:

$$\dot{K}=I_K-\delta_k K, \tag{1.3}$$

where I_K is physical capital investment and δ_k is depreciation rate of physical capital. The constraint for resource is given as:

$$Y=C+I_K+I_H, \tag{1.4}$$

where I_H, is the investment made on human capital stock and $I_H=G$ where G denote public expenditures. We assume that the government sets its tax rate as a fixed fraction of output,τ , i.e.

$$I_H=\tau Y=G,\ 0<\tau<1. \tag{1.5}$$

Therefore government finances its expenditure on human capital through income tax revenue that is collected from private agents in the economy. **Following recent literature (Dioikitopoulos and Kalyvitis, 2015; Turnovsky,**

[1]A further analysis of utility preferences can be performed for $\gamma=1-\nu$ case.

2000), the government expenditure is represented as a fixed proportion of output with a flat tax rate. This tax revenue is used to finance human capital. The government activity is endogenously determined. We are assuming that human capital is a public good; it is the government who is providing education under the constraint of balanced budget. Combining equation (1.3) for physical capital and equations (1.4) and (1.5), we get the following equation for the capital stock:

$$\dot{K} = (1-\tau)Y - C - \delta_k K. \tag{1.6}$$

The final form of evolution equation for physical capital is

$$\dot{K} = (1-\tau)K^{\alpha}H^{1-\alpha} - C - \delta_k K, \ 0 < \tau < 1, \ K(0) = K_0. \tag{1.7}$$

The accumulation of physical capital depends upon the income after taxation, consumption by agents and the physical capital after depreciation.

1.1.4 Human capital

The equation of motion for human capital is:

$$\dot{H} = \tau\phi Y S - \delta_h H, \tag{1.8}$$

where ϕ is a constant efficiency parameter, δ_h is the human capital depreciation and S is a variable for average living standard. This sort of specification is often employed in the growth literature for models with human and health capital (see e.g.Hosoya (2016)). We assume $S = (K/H)^{\epsilon}$ and then equation (1.8) takes following form:

$$\frac{\dot{H}}{H} = \tau\phi\left(\frac{K}{H}\right)^{\alpha+\epsilon} - \delta_h, \tag{1.9}$$

where ϵ represents the degree of externality.

A concrete index (S) is used to measure the living standard. The idea is that economies with sufficient levels of human capital have higher standards of living. In our model there are human capital related externalities where individuals experience increasing social returns to human capital either through learning by doing or research and development. This in turn increases the accumulation of human capital in the economy. So, unlike the production function type of externality, for our case we are assuming these type of positive externalities of human capital. An economy that has a higher score on index translates into higher living standards. In my model, human capital is considered to be a very valuable kind of capital. The human capital is as important as physical capital in generating a successful economy. The immediate effect of human capital is raising the skill level of workers. Human capital also helps in the efficient allocation of resources in society. Physical capital and human skills are considered complementary. This could be explained by looking at the new investment in a country; As the country develops it brings in physical capital and it also requires skilled workers to operate it. Literature also shows that its not only machinery that needs advancement but we need to invest in human capital to have economic progress (Burton A. Weisbrod, 1962).

1.2 Endogenous time preferences assumptions

We assume that $\rho(C, H)$ satisfies following assumptions:

Assumption 1: $\rho(C, H) \geq m > 0$

According to assumption 1 there is an existence of lower bound for time preference rate denoted by m and the rate if time preference is strictly positive.

Assumption 2: $\rho_C \geq 0$ and $\rho_H \leq 0$.

Assumption 2 explains the relationship between consumption and human capital with respect to time preference rate. Time preference is positively related to the aggregate level of consumption and negatively related to aggregate human capital. This follows the literature patterns which has confirmed the linkage between time preference rate and social factors (see e.g., Epstein & Hynes (1983); Schumacher (2009); Choi et. al. (2008); Dioikitopoulos and Kalyvitis (2015); Agénor (2010)). There is evidence that as the consumption level goes up, people become impatient. Similarly, if the human capital is higher in an economy then individuals will be patient and will forgo current consumption for future consumption Becker and Mulligan (1997).

Assumption 3: $\rho(C, H) = \rho(C/H)$ and $\rho'(.) \geq 0$.

Assumption 3 states that the rate of time preference is defined as the ratio of consumption and human capital. **This assumption also implies linearly homogeneity.** The rate of time preference is bounded at a steady state (Palivos & Zhang, 1997).

1.3 The representative agent problem

There is a finite number of agents who live for an infinite time period. The agents in the economy are identical and all variables are continuous as well as differentiable functions of time period t. The utility function is a non-separable in terms of consumption and stock of human capital. The representative agent lives from period 0 until forever and discounts the future at the rate $\rho > 0$. The optimization problem is maximize

$$\int_0^{\infty} U(C, H) \exp\left[-\int \rho(C, H)\right] dt, \tag{1.10}$$

subject to constraint

$$\dot{K} = (1-\tau)K^{\alpha}H^{1-\alpha} - C - \delta_k K,\ 0 < \tau < 1,\ K(0) = K_0. \tag{1.11}$$

and H is exogenously determined from **(1.9).**

1.3.1 Solving the model around the equilibrium

The present value Hamiltonian function for the model is

$$H = \frac{(C^{\nu}H^{\gamma})^{1-\sigma} - 1}{1-\sigma} e^{-\Delta(t)} + \lambda[(1-\tau)K^{\alpha}H^{1-\alpha} - C - \delta_k K] \tag{1.12}$$

where $\Delta(t) = \int \rho(C,H)dt$, taking time derivative of $\Delta(t)$ we get: $\dot{\Delta}(t) = \rho(C,H)$.

The control variable is $C(t)$ because representative agents have control over consumption today and tomorrow, $K(t)$ is state variable that is not controlled completely and λ is costate variable.

After taking the first order conditions (See Appendix A) and solving them we are able to get the differential equations of this model given as:

$$\frac{\dot{K}}{K} = (1-\tau)K^{\alpha-1}H^{1-\alpha} - \frac{C}{K} - \delta_k, \tag{1.13}$$

$$\frac{\dot{C}}{C} = \frac{1}{\nu(1-\sigma)-1}[(-\alpha(1-\tau)(\frac{K}{H})^{\alpha-1}$$
$$+\delta_k - \gamma(1-\sigma)\tau\phi(\frac{K}{H})^{\alpha+\epsilon} + \gamma(1-\sigma)\delta_h + \rho(\frac{C}{H})], \tag{1.14}$$

$$\frac{\dot{H}}{H} = \tau\phi\left(\frac{K}{H}\right)^{\alpha+\epsilon} - \delta_h. \tag{1.15}$$

It is important to mention here that in dynamical system the externality parameter is involved in the model through exogenously

determined human capital H. By taking H as exogenous, we mean that we are not taking it as a choice variable in the model and we have only one co-state variable λ for physical capital constraint. This is the standard procedure followed by recent literature (Dioikitopoulos and Kalyvitis, 2015; Hosoya, 2016; Gaspar et. al., 2014).
For our case we don't require the joint concavity condition of utility function on C and H. Utility function is concave in C but not in H as it is exogenous. The constraint for K is concave in C and K. The Mangasarian (1966) sufficiency theorem holds.

We can not solve the model for original variables. The BGP can be derived for our model by defining two new variables as: $X = C/K$, and $Z = K/H$. The above system of three differential equations can be rewritten as a system two differential equations in terms of X and Z. (See Appendix A)

$$\frac{\dot{X}}{X} = -(1-\tau)Z^{\alpha-1}[1 + \frac{\alpha}{\nu(1-\sigma)-1}] - \frac{\gamma(1-\sigma)\tau\phi Z^{\alpha+\epsilon}}{\nu(1-\sigma)-1}$$
$$+\frac{\rho(.)}{\nu(1-\sigma)-1} + X + [\frac{\gamma(1-\sigma)\delta_h}{\nu(1-\sigma)-1} + \frac{\delta_k}{\nu(1-\sigma)-1} + \delta_k], \tag{1.16}$$

$$\frac{\dot{Z}}{Z} = (1-\tau)Z^{\alpha-1} - X - \tau\phi Z^{\alpha+\epsilon} + \delta_h - \delta_k. \tag{1.17}$$

1.3.2 The decentralized equilibrium

Now we move on to look at the decentralized equilibrium and balanced growth path (BGP). Decentralized equilibrium is defined as:

Definition: "*In the decentralized equilibrium, C, K, H and Y grow at the same constant growth rate and therefore X and Z are constants.*" In our case we can denote steady state as: $g = \frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{H}}{H} = \frac{\dot{Y}}{Y}$, $X = X^*$, $Z = Z^*$. where Z satisfies this equation:

$$0 = -\frac{(1-\tau)\alpha}{\nu(1-\sigma)-1}Z^{\alpha-1} - \tau\phi Z^{\alpha+\epsilon}\big[1+\frac{\gamma(1-\sigma)}{\nu(1-\sigma)-1}\big]$$
$$+\frac{\rho(.)}{\nu(1-\sigma)-1} + \delta_h\big[1+\frac{(\gamma(1-\sigma))}{\nu(1-\sigma)-1}\big] + \frac{\delta_k}{\nu(1-\sigma)-1} \tag{1.18}$$

where X is given as:

$$X = (1-\tau)Z^{\alpha-1} - \phi\tau Z^{\alpha+\epsilon} + \delta_h - \delta_k > 0 \tag{1.19}$$

We are assuming the non negativeness condition of X, where $X > 0$.

Defining two new functions Γ and Ψ as:

$\Gamma(Z) : \frac{\alpha(1-\tau)Z^{\alpha-1}}{1-\nu(1-\sigma)} - \tau\phi Z^{\alpha+\epsilon}[1-\frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}] + \delta_h[1-\frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}] - \frac{\delta_k}{1-\nu(1-\sigma)}$
$\Psi(Z) = \frac{\rho(XZ)}{1-\nu(1-\sigma)}$.

We will end up having two different scenarios.

Case 1: Existence of unique physical to human capital ratio.

Case 2: Existence of multiple physical to human capital ratio.

It is difficult to analyze analytical properties of functions Γ and Ψ in general form, therefore we will use numerical simulation analysis to investigate the cases of unique and multiple equilibrium.

1.4 Numerical simulations

This section looks at the numerical simulations results. This would help us to explore the results derived in the previous section. Firstly we will determine the benchmark value of all the parameters involved in our model.

It is difficult to determine the values for the benchmark parameter for numerical simulations. For this, we look at the literature for the values being used in other papers.

The share of physical capital is represented by α. In literature the value of α

used is 0.35 by Hosoya (2016). Depreciation of human and physical capital is often taken equal in literature and close to zero, Dioikitopoulos & Kalyvitis (2010). ϕ is the constant efficiency parameter for human capital accumulation and in the literature, authors have used a value as 0.1 Hosoya (2016).
The discount factor is represented by m. In the literature, there is a range of values for the discount factor used. Some have used a discount factor m as low as 0.001, Antoci et al. (2011) while other authors have used a range from 0.05 to 0.08 [Carboni & Russu (2013); Ortigueira and Santos (1997) ; Ladrn-de-Guevara et.al (1997)]. Itaya (2008) and Fernndez et al. (2012) have used a discount factor of 0.045 and 0.2 respectively.

σ represents the inverse of the intertemporal elasticity of substitution. There also exists a range of σ in literature. Hosoya (2012) has used 0.32 σ whereas Ladrn-de-Guevara et.al (1997) have used 0.7 value of σ in his analysis. On the other hand Carboni & Russu (2013) use 0.01 which is a low value of the inverse of the intertemporal elasticity of substitution. There also exists higher values of the inverse of the intertemporal elasticity of substitution in literature. Itaya (2008) have used $\sigma = 2$ whereas Fernndez et al. (2012) has taken $\sigma = 2.25$ and Hosoya (2014) uses a value 1.2 for σ.
Tax rate τ is also in line with the literature values being used for the analysis. Hosoya (2012) used 0.04 as the tax rate whereas in another paper Hosoya (2014) used a value of 0.05 for numerical simulations. **All of these parameter values are in line with literature and references are provided with each parameter value chosen for the model. Note that the justification for this relatively low tax rate is that for our model, we are assuming that the government activity is to provide education to individuals in the economy.** Externality ϵ for human capital is assigned a value of 0.15 value in literature,Hosoya (2016).

The table (1.1) shows the benchmark case for all the parameters.

Table 1.1: Benchmark parameters

α	0.4
δ_h	0.003
δ_k	0.01
b	0.1
m	0.005
ϕ	0.15
σ	0.5
ϵ	0.1
ν	1
γ	3.5
τ	0.03

The form of time preference is taken as:

$$\hat{\rho} = b(\frac{C}{H}) + m \tag{1.20}$$

b is the slope if impatience function which is a function and consumption and human capital.

1.4.1 Multiple equilibria with effect of change in parameters

Once we have come up with the benchmark values of parameters involved in the model, the next step is to use this set of parameters and look at how changing different parameter values would impact the low and high growth rates. Table (1.2) shows how changing values of γ, b, σ, ϵ would change g_L and g_H and the growth gap shrinkage is observed.

The weight that agents put on human capital in utility function is

Table 1.2: Effect of change in parameters

	g_L	g_H
	0.0059	0.0432
$\gamma = 1.5$	0.0058	0.0449
$b = 0.2$	0.0032	0.0445
$\sigma = 0.1$	0.0063	0.0400
$\epsilon = 0.05$	0.0053	0.0376

represented by γ . The existence of unique or multiple equilibria depends upon the value of gamma. For $\gamma \leq 0.68$ there exist a unique equilibrium for our model, whereas for $\gamma > 0.68$ there exist multiple equilibria: low growth and high growth. The benchmark value of gamma for our analysis is 3.5 .**It is important to note here that the uniqueness or multiplicity of the model is dependent on the values of γ (weight assigned to human capital) in our model, taking into account the other benchmark parameters.**
In order to see the impact of change in γ all other parameter values same except γ. The result of the numerical simulation is shown in the table (1.2). If we increase the value of γ from 1.5 to 3.5 the growth rate changes. An increase in γ leads to a decrease in high growth rate g_H from 0.0449 to 0.0432, whereas low growth g_L increases from to 0.0058 to 0.0059. This results in shrinking the gap from 0.0391 to 0.0373

By comparing two countries preferences on human capital, we come across an interesting result where individuals who put in more value on human capital($\gamma = 3.5$) in their utility function would be better off as compared to those who put in less weight on human capital ($\gamma = 1.5$). This increases g_L and decreases g_H hence lowering the gap between the two growth rates.

The figure (1.1) below shows the gap shrinkage between low growth and high growth equilibrium that is resulted from an increase in value of γ.

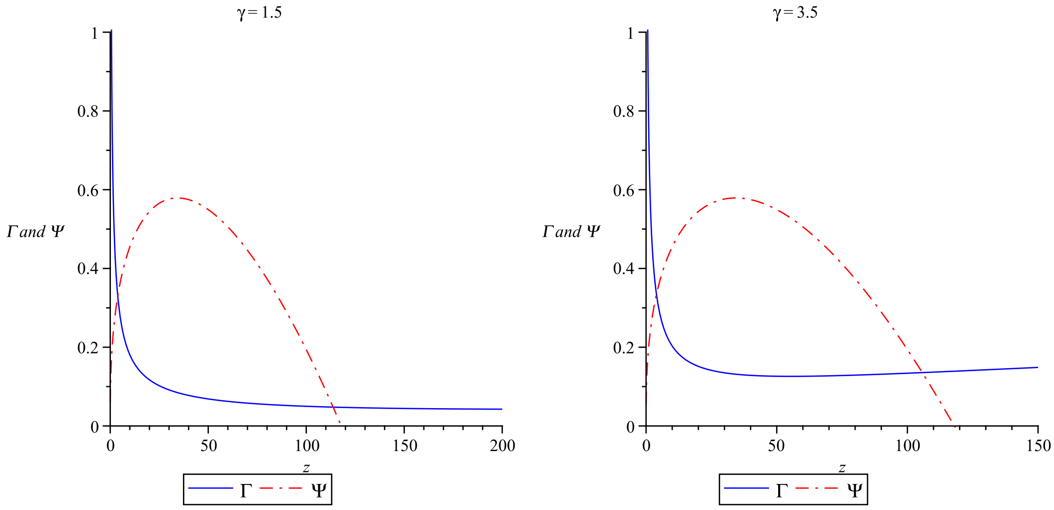

Figure 1.1: Effect of change in γ on low and high growth equilibria when $\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b= 0.1,m= 0.005, $\phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \tau = 0.03$

b is the slope of the impatience function. The benchmark value of b is 0.1. The results presented in table 1.2 shows that a decrease in b from 0.2 to 0.1 shrinks the gap between low and high equilibrium. As b changes from 0.1 to 0.2, g_L increases from 0.0032 to 0.0059 ,whereas g_H decreases from 0.0445 to 0.0432. Since a decrease in b from 0.2 to 0.1 means individuals become more patient, individuals are consuming less and saving more. This would help the economy with a lower b to increase g_L and decrease g_H thus lowering the gap between two equilibria.

The figure (1.2) shows that growth gap is shrinking due to decreases in parameter b.

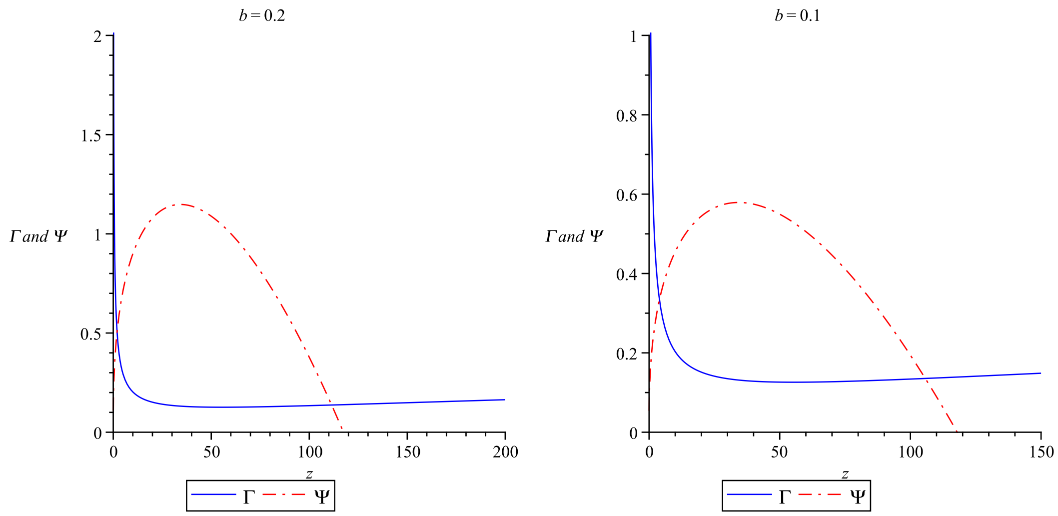

Figure 1.2: Effect of change in b on low and high growth equilibria when $\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,m$= 0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \gamma = 3.5, \tau = 0.03$

The inverse intertemporal elasticity of substitution is σ. This section looks at how changing σ affect the gap between g_L and g_H. All other parameter values are the same. The numerical simulation result presented in table (1.2) shows that a decrease in σ from 0.5 to 0.1 shrinks the gap between the low and high growth equilibriua. The low growth g_L increases from 0.0059 to 0.0063 whereas, the high growth decreases from 0.0432 to 0.0400. The gap is decreased from 0.0373 to 0.0337 .

The figure (1.3) shows that a change in σ results in shrinking the gap between the two equilibria.

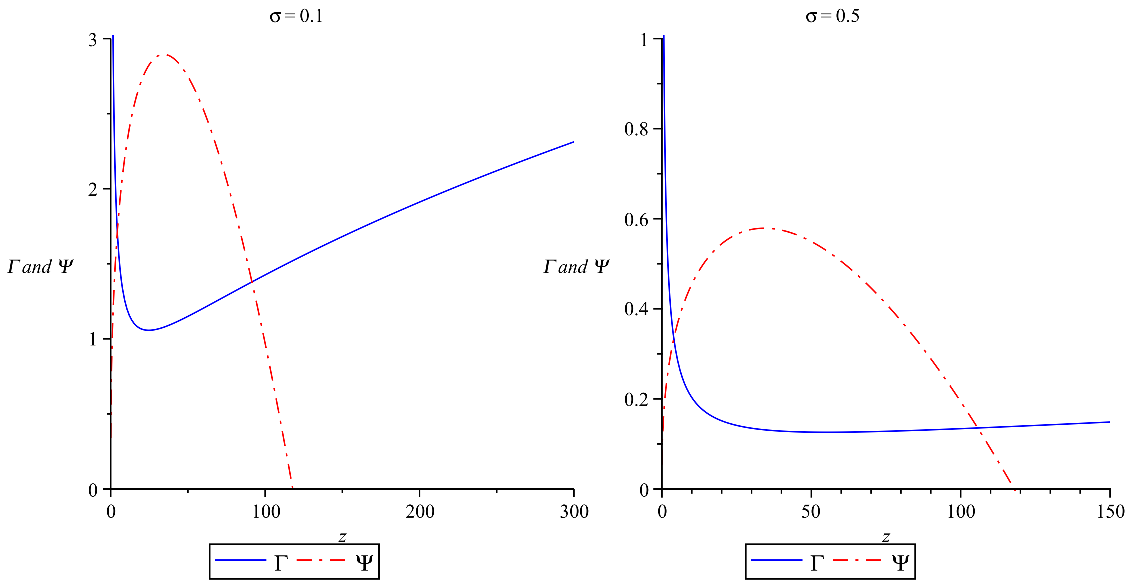

Figure 1.3: Effect of change in σ on low and high growth equilibria when $\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b= 0.1,m= 0.005, $\phi = 0.15, \epsilon = 0.1, \nu = 1, \gamma = 3.5, \tau = 0.03$

In order to look at the impact of ϵ, we keep all the parameter values the same as stated and only change the value of epsilon ϵ. The interesting result over here shows that due to the externality effect both g_L and g_H increases. This pattern is observed because we are also taking into account endogenous time preferences. More patient individuals invest more time in the accumulation of human capital due to which the external effect increases both equilibria. An increase in ϵ from 5 % to 10 % increases the low growth equilibrium from 0.0053 to 0.0059. Similarly, the high growth increases from 0.0376 to 0.0432 .

Figure (1.4) shows the multiple equilibria under the change in degree of externality ϵ

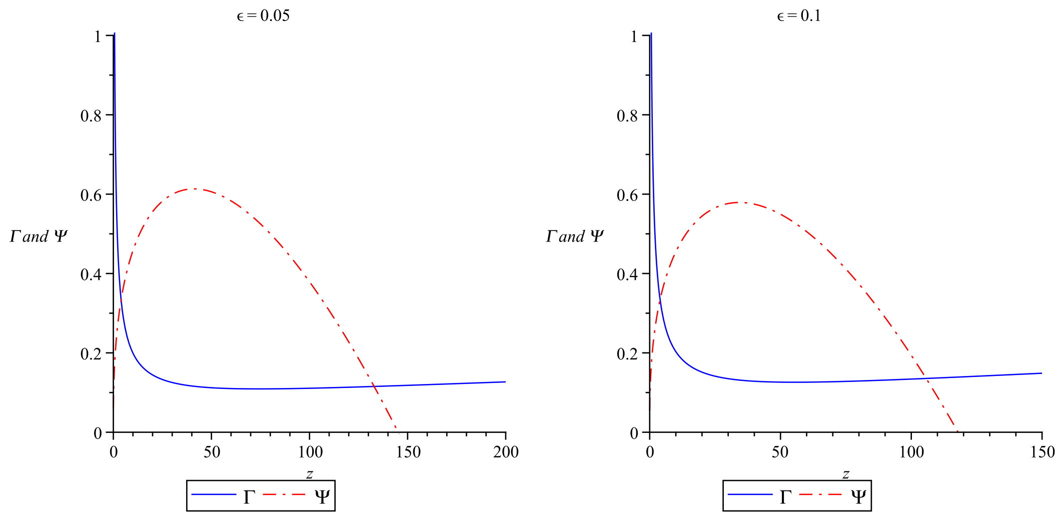

Figure 1.4: Effect of change in ϵ on low and high growth equilibria when $\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b= 0.1,m= 0.005, $\phi = 0.15, \nu = 1, \sigma = 0.5, \gamma = 3.5, \tau = 0.03$

1.5 Conclusion

This chapter mainly looks at the basic framework of the economic growth model which includes human capital externalities along with endogenous time preferences. The model is solved along BGP.

The benchmark value for parameters was taken from literature. This chapter explored how changing parameter values (α, b, γ, ϵ) through numerical simulations can lead to shrinking the gap between low and high growth rates. Change in all parameters values, except for ϵ result in shrinking the gap between the low and high growth rates. The low growth rate g_L was increasing whereas the high growth rate equilibrium g_H was decreasing.

A closer look at σ shows that as individuals value human capital more they would prefer to invest more in human capital that would increase the accumulation of human capital thus moving the economy to a high growth path. A similar type of behavior is depicted by b, where a decrease in b makes individuals more patient and encourages them to save and invest in human capital thus moving economy towards higher steady-state growth. Furthermore a decrease in σ also resulted in a shrinking gap as individuals value future more and save more. An increase in the degree of externality ϵ resulted in increasing the high and low growth rate equilibrium due to joint impact of positive externalities and endogenous time preferences.

The next thing to look up is the stability of equilibrium. Among the multiple equilibria. We test whether the low growth equilibrium and high growth equilibrium are stable paths for the model. The next chapter explores the meaningfulness of each equilibrium and how we can improve the growth rate for stable economic growth.

Chapter 2

Stability analysis of a macroeconomic model with human capital externalities

Once we have come up with the existence of multiple equilibria in our model the next step is to determine which of the two equilibria is meaningful. To check this we need to perform a stability analysis for our model that would help to determine which equilibrium is stable and how to improve that equilibrium.

In this chapter, we analyze the stability and transitional dynamics around equilibrium.

2.1 Stability and transitional dynamics around equilibrium

For the analysis, we will use the two-dimensional system of equations (1.16) and (1.17) respectively. The Jacobian matrix is evaluated at the optimal steady-state values of control variable X and state variable Z involved in the model.

$$J\mid_{(X^*,Z^*)}= \begin{bmatrix} \frac{\partial \dot{X}}{\partial X}\mid_{(X^*,Z^*)} & \frac{\partial \dot{X}}{\partial Z}\mid_{(X^*,Z^*)} \\ \frac{\partial \dot{Z}}{\partial X}\mid_{(X^*,Z^*)} & \frac{\partial \dot{Z}}{\partial Z}\mid_{(X^*,Z^*)} \end{bmatrix}, \tag{2.1}$$

where the Jacobian Matrix for the model is given as following expression (2.2) below:

$$J^* = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{2.2}$$

where a_{11} is given by

$$a_{11} = X + \frac{XZ\rho'(XZ)}{\nu(1-\sigma)-1}, \tag{2.3}$$

a_{12} is given by

$$\begin{aligned} a_{12} &= (\alpha-1)(1-\tau)[1+\frac{\alpha}{\nu(1-\sigma)-1}]XZ^{\alpha-2} \\ &-\frac{g(\alpha+\epsilon)(1-\sigma)\phi\tau XZ^{\alpha+\epsilon-1}}{\nu(1-\sigma)-1} + \frac{\rho'(XZ)X^2}{\nu(1-\sigma)-1} \end{aligned} \tag{2.4}$$

a_{21} and a_{22} are:

$$a_{21} = -Z, \tag{2.5}$$

$$a_{22} = (\alpha-1)(1-\tau)Z^{\alpha-1} - (\alpha+\epsilon)\tau\phi Z^{\alpha+\epsilon}. \tag{2.6}$$

Hence, following is then the linear system for the model:

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} X + \frac{XZ\rho'(XZ)}{\nu(1-\sigma)-1} & a_{12} \\ -Z^* & (\alpha-1)(1-\tau)Z^{\alpha-1} - (\alpha+\epsilon)\tau\phi Z^{\alpha+\epsilon} \end{bmatrix} \begin{bmatrix} X - X^* \\ Z - Z^* \end{bmatrix}. \tag{2.7}$$

There are three different cases to check the stable equilibrium through the signs of Jacobian and trace. The following table (2.1) summarizes the criteria for stability.

Table 2.1: Stability Criteria through Jacobian & Trace

$J^* < 0$	Saddle-path stable equilibrium
$J^* > 0$ *and* $T^* > 0$	Unstable equilibrium
$J^* > 0$ *and* $T^* < 0$	Stable equilibrium

The trace, $tr(J^*)$ is given by equation (2.8):

$$\begin{aligned} tr(J^*) = X^* + \frac{\rho'(.)X^*Z^*}{\nu(1-\sigma)-1} \\ +(\alpha-1)(1-\tau)Z^{*\alpha-1} - (\alpha+\epsilon)\tau\phi Z^{*\alpha+\epsilon}, \end{aligned} \tag{2.8}$$

And the determinant of the Jacobian matrix (detJ) is given by equation (2.9):

$$
\begin{aligned}
detJ^* = \Bigg[\frac{X^*Z^*b[-\phi\tau(\alpha+\epsilon+1)Z^{*\alpha+\epsilon}+\alpha(1-\tau)Z^{*\alpha-1}+\delta_h-\delta_k]}{\nu(1-\sigma)-1}\Bigg] \\
+X^*Z^*\big[-\phi\tau(\alpha+\epsilon)Z^{*\alpha+\epsilon-1}+\frac{\alpha(1-\alpha)(1-\tau)Z^{*\alpha-2}}{\nu(1-\sigma)-1} \\
-\frac{\gamma\phi\tau(1-\sigma)(\alpha+\epsilon)Z^{*\alpha+\epsilon-1}}{\nu(1-\sigma)-1}\big]
\end{aligned} \tag{2.9}
$$

In order to look at the signs of Jacobian and trace we will use numerical simulation analysis as followed by [Chaudhry et. al. (2017); Hosoya (2012); Dioikitopoulos & Kalyvitis (2010) and Xie (1994)].

2.2 Numerical simulations

In this section, we analyze the unique and multiple equilibria cases. We will perform similar analysis as done in chapter 1 for both unique and multiple cases. In addition to looking at the growth rates for low and high steady state we will also look at the important ratios such as consumption to human capital (C/H), consumption to GDP ratio (C/Y), and private investment as a ratio of GDP ($\dot{K}/Y$).

2.2.1 Equilibrium properties under unique equilibrium

As presented in chapter 1 there exist unique and multiple growth paths depending upon the value of γ weight assigned to human capital. For $\gamma \leq 0.68$ there exists unique steady-state equilibrium whereas for $\gamma > 0.68$ there is an existence of two equilibria i-e low and high steady state growth paths for the benchmark values. The table (2.2) presents the numerical simulation result under the unique case. For this, all the parameter values are the same as in benchmark case except for γ which is taken as 0.4

Table 2.2: Equilibrium properties under unique equilibrium
$\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.1,m= $0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \gamma = 0.68, \tau = 0.03$

	Low growth equilibrium
Growth rate	0.0057
$X^*(C/H)$	0.4229
$Z^*(K/H)$	3.7534
Consumption/Human capital(C/H)	1.5874
Consumption/GDP(C/Y)	0.9352
Private investment/GDP($\dot{K}/Y$)	0.01264
$\hat{\rho}$	0.1637
$detJ^*$	$-.1394$
$tr(J^*)$	-0.1621
$S = Z^\epsilon$	1.1414
Local property	Saddle path stable

The unique low growth rate is 0.0057, whereas the rate of time preference is 0.16 which means that agents are less patient and will consume more. The consumption as a ratio of GDP is 0.9352 . In order to look at the stability we examine the signs of the jacobian and trace. This is a saddle path stable equilibrium. Figure (2.1) shows the existence of unique steady state equilibrium for value of γ= 0.4 ceteris paribus. The function Ψ and Γ intersects only at one point.

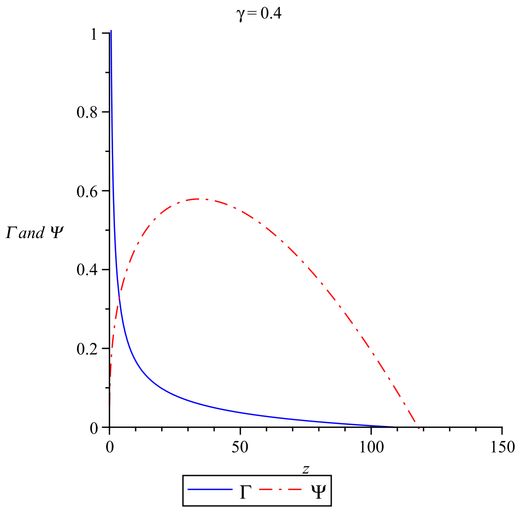

Figure 2.1: Equilibrium properties under unique equilibrium for $\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.1,m= $0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \gamma = 0.4, \tau = 0.03$

2.2.2 Equilibrium properties under multiple equilibrium

For the value of $\gamma > 0.68$, keeping all other parameter values same we will have multiple equilibria. Table (2.3) shows the numerical simulation results for $\gamma = 3.5$ case.

The growth rate for low equilibrium case is 0.0059 and 0.0432 for the high growth. The rate of time preference is 0.16 (low growth case) which is higher than in the high growth time preference rate. Whereas the degree of externality is higher in the high growth equilibrium as compared to low growth equilibrium. For the low growth case, the degree of externality is 1.147 which is less than that of the high growth equilibrium (1.5936). This turns out to be a society that follows a high growth path with greater external effects of human capital and standard of living is improved.

The low growth equilibrium is saddle path stable as $det(J^*) < 0$ and

$tr(J^*) < 0$. The high growth equilibrium is stable as $det(J^*) > 0$ and $tr(J^*) < 0$.

Table 2.3: Equilibrium properties under multiple equilibrium

$\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.1,m= $0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \gamma = 3.5, \tau = 0.03$

	Low growth equilibrium	High growth equilibrium
Growth rate	0.0059	0.0432
$X^*(C/K)$	0.4080	0.0059
$Z^*(K/H)$	3.9717	105.69
Consumption/Human capital (C/H)	1.6206	0.6276
Consumption/GDP (C/Y)	0.9334	0.0972
Private investment/GDP($\dot{K}/Y$)	0.0136	0.7088
$\hat{\rho}$	0.1670	0.0677
$detJ^*$	-0.1268	0.0067
$tr(J^*)$	-0.1749	-0.1782
$S = Z^{\epsilon}$	1.1478	1.5936
Local property	Saddle path stable	Stable

Figure (2.2) shows the graph for multiple equilibrium. It is clearly shown that the functions Ψ and Γ intersects both at two points. This shows the existence of two equilibria: low and high equilibrium.

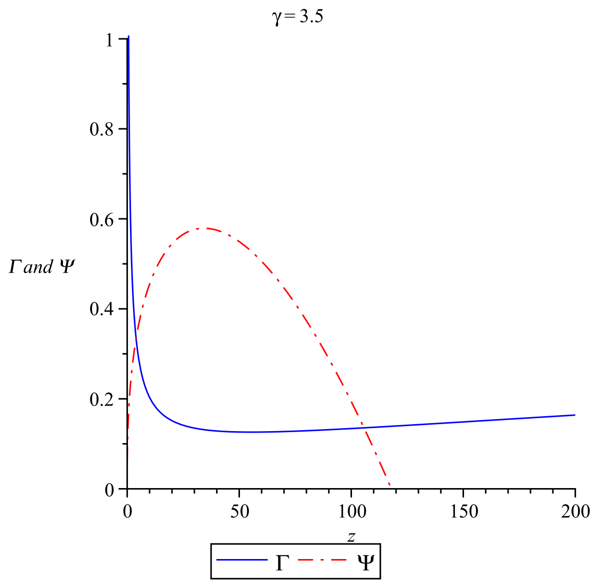

Figure 2.2: Equilibrium properties under multiple equilibrium for $\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.1,m= $0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \gamma = 3.5, \tau = 0.03$

2.2.3 Multiple equilibria and changes in inverse of intertemporal elasticity of substitution

In this section we observe the changes in the equilibrium properties and the stability of the equilibrium because of changes in σ, keeping all other parameters same. σ is the inverse of intertemporal elasticity of substitution.

When σ decrease from 0.5 to 0.1 growth gap shrinks. The low growth increases from 0.59% to 0.63%. Whereas, high growth decreases from 4.32% to 4%. In low growth case, the main driving factor for increasing growth is a higher private investment to GDP ratio. Private investment as a ratio of GDP is increasing from 0.0136 to 0.0150 At the same time, agents are consuming less. For $\sigma = 0.5$ the consumption to GDP is 0.933 while it decreases to 0.9310 for $\sigma = 0.1$ case. In the high growth case, the individuals are less patient; they consume more and invest less therefore the high growth equilibrium decreases. The increase in low growth and a decrease in high growth shrinks the gap between the two growth rates. Table (2.4) shows numerical simulation result foe change in σ.

Table 2.4: Effect of change in σ

$\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.1,m= $0.005, \phi = 0.15, \sigma = 0.1, \epsilon = 0.1, \nu = 1, \gamma = 3.5, \tau = 0.03$

	Low growth equilibrium	High growth equilibrium
Growth rate	0.0063	0.0400
$X^*(C/K)$	0.3893	0.0145
$Z^*(K/H)$	4.2762	91.459
Consumption/Human capital(C/H)	1.6648	1.3293
Consumption/GDP(C/Y)	0.9310	0.2183
Private investment/GDP($\dot{K}/Y$)	0.0150	0.6014
$\hat{\rho}$	0.1714	0.1379
$detJ^*$	-0.5590	0.0680
$tr(J^*)$	-1.5235	-1.3750
$S = Z^\epsilon$	1.1563	1.5708
Local property	Saddle path stable	Stable

2.2.4 Multiple Equilibria and changes in slope of impatience function

b represents the slope of impatience function. Table (2.5) shows the numerical simulation results for change in b.

A decrease in b from 0.2 to 0.1 reduces the gap between the high and low growth equilibria. In the low growth situation growth is driven mainly because of agents being more patient who consume less and invest more. Whereas g_H decreases because consumption is increasing and investment is decreasing due to agents becoming less patient.

Table 2.5: Effect of change in b

$\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.2,m= $0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \gamma = 3.5, \tau = 0.03$

	Low growth equilibrium	High growth equilibrium
Growth rate	0.0032	0.0445
$X^*(C/K)$	0.6352	0.0028
$Z^*(K/H)$	1.9561	111.4
Consumption/Human capital(C/H)	1.2426	0.3178
$Consumption/GDP(C/Y)$	0.9501	0.0482
Private investment/GDP($\dot{K}/Y$)	0.0049	0.7526
$\hat{\rho}$	0.2535	0.0685
$detJ^*$	-0.3135	0.0071
$tr(J^*)$	-0.2540	-0.1824
$S = Z^\epsilon$	1.0694	1.6021
Local property	Saddle path stable	Stable

2.2.5 Multiple equilibria and changes in weight assigned to human capital

The weight assigned to human capital by agents in the utility function is γ. The benchmark value for γ is 3.5. An increase in gamma from 1.5 to 3.5 shrinks the growth gap. The consumption by individuals is going up. In this case, high growth is decreasing mainly due to agents becoming less patient. The rate of time preference is increasing from 0.023 to 0.067 and consumption as a ratio of GDP is increasing from 0.0049 to 0.0972 .The degree of externality has increased from 1.143 to 1.147

Table 2.6: Effect of change in γ

$\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.1,m= $0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.1, \nu = 1, \gamma = 1.5, \tau = 0.03$

	Low growth equilibrium	High growth equilibrium
Growth rate	0.0058	0.0449
$X^*(C/K)$	0.4177	0.0016
$Z^*(K/H)$	3.8269	113.73
Consumption/Human capital(C/H)	1.5987	0.1890
$Consumption/GDP(C/Y)$	0.9346	0.0284
Private investment/GDP($\dot{K}/Y$)	0.012	0.7703
$\hat{\rho}$	0.1648	0.0239
$detJ^*$	-0.1350	0.0021
$tr(J^*)$	-0.1665	-0.0941
$S = Z^\epsilon$	1.1436	1.6054
Local property	Sadle path stable	Stable

2.2.6 Multiple Equilibria and changes in degree of externality

The degree of externality is denoted by ϵ. When there is increase in the degree of the externality, the low and high growth both increases with the low growth increasing from 0.53 % to 0.59%. Also, the high growth equilibrium is increasing from 3.76 % to 4.32%. Table (2.7) presents the numerical simulation results for ϵ.

This is an interesting change to be observed for this specific parameter in the model. For all other parameter changes, there was a shrinking gap between low and high equilibrium. But for the effect of epsilon, we observe the opposite case to be occurring. This is due to high C/H in both cases and a higher investment by agents. The standard of living also improves from 1.01 to 1.14 in the low growth case and from 1.27 to 1.59 in the high growth.

Table 2.7: Effect of change in ϵ

$\alpha = 0.4, \delta_h = 0.003, \delta_k = 0.01$,b=0.1,m= $0.005, \phi = 0.15, \sigma = 0.5, \epsilon = 0.05, \nu = 1, \gamma = 3.5, \tau = 0.03$

	Low growth equilibrium	High growth equilibrium
Growth rate	0.0053	0.0376
$X^*(C/K)$	0.4102	0.0039
$Z^*(K/H)$	3.9469	132.93
Consumption/Human capital(C/H)	1.6192	0.5267
$Consumption/GDP(C/Y)$	0.9350	0.0744
Private investment/GDP($\dot{K}/Y$)	0.0121	0.7074
$\hat{\rho}$	0.1669	0.0576
$detJ^*$	-0.1289	0.0048
$tr(J^*)$	-0.1727	-0.1506
$S = Z^\epsilon$	1.0710	1.2769
Local property	Sadle path stable	Stable

2.3 Conclusion

In this chapter, the stability of the equilibrium of our model is analyzed. This is an important result since it would be helpful in explaining the growth differences between countries. This could be illustrated by the effect of the change

of parameter involved in our model that includes: (γ), (σ), b and (ϵ). In all cases, we observed the growth gap shrinking between the two equilibria except for epsilon. The impact of these changes was also observed on important ratios used in literature. The economy was converging to a high growth steady state equilibrium for all cases as this was a stable path.
A decrease in σ results in shrinking the gap between low and high equilibrium. A decrease in σ resulted in an increase in the low growth rate which is due to agents becoming more patient and they are willing to forgo current consumption and make more investments. On the other hand, the high growth rate decreases due to individuals becoming less patient.
A similar pattern is observed for the slope of impatience function. As b decreases growth gap shrinks and economy converges to high growth equilibrium where individuals are more patient. They would be willing to invest and reduce current consumption which has a positive impact for growth of the economy.
Furthermore γ is an important factor since an increase in gamma is shrinking the growth gap. An increase in the degree of externality results in increasing the high and low growth equilibria and the standard of living is also improved for both cases. The human capital externality is helpful for economies to get out of a poverty trap since human capital accumulation would be higher which increases the productivity and growth in the economy.

Table 2.8: Effect of change in parameters on low growth equilibrium

	g_L	X_L	Z_L	$(C/Y)_L$	$(K/Y)_L$	ρ_L	Z_L^ϵ
$\sigma \downarrow$	(+)	(−)	(+)	(−)	(+)	(+)	(+)
$\gamma \uparrow$	(+)	(−)	(+)	(−)	(+)	(+)	(+)
$b \downarrow$	(+)	(−)	(+)	(−)	(+)	(−)	(+)
$\epsilon \uparrow$	(+)	(−)	(+)	(−)	(+)	(+)	(+)

Table 2.9: Effect of change in parameters on high growth equilibrium

	g_H	X_H	Z_H	$(C/Y)_H$	$(K/Y)_H$	ρ_H	Z_H^ϵ
$\sigma \downarrow$	$(-)$	$(+)$	$(-)$	$(+)$	$(-)$	$(+)$	$(-)$
$\gamma \uparrow$	$(-)$	$(+)$	$(-)$	$(+)$	$(-)$	$(+)$	$(-)$
$b \downarrow$	$(-)$	$(+)$	$(-)$	$(+)$	$(-)$	$(-)$	$(-)$
$\epsilon \uparrow$	$(+)$	$(+)$	$(-)$	$(+)$	$(-)$	$(+)$	$(+)$

Chapter 3

Fiscal implications

This chapter explores the fiscal policy implications resulting from the government. This is done by endogenously determining public policy. Since there exist multiple equilibria, the government can possibly play a role by choosing tax level which would change the preferences of agents. Given the competitive decentralized equilibrium (CDE) the government determines how much funds are needed to finance the optimal growth of human capital in the economy.

Distortionary income tax revenues are helpful since they provide government with more tax revenues. In our case increased taxation has a counter balancing effect. With an increase in the tax rate, the individuals would become impatient and consumption would be higher but through increased tax revenues, government would increase the public expenditure on education that in turns boost the growth rate in the economy.

3.1 Growth maximizing fiscal policy

Growth maximizing fiscal policy is defined as:

"A growth-maximizing (GM) allocation is given when

(i) the government chooses the tax rate and aggregate allocations

in order to maximize the growth rate of the economy by taking into account the aggregate optimality conditions of the CDE, and (ii) the government budget constraint and the feasibility and technological conditions are met." (Dioikitopoulos & Kalyvitis, 2010)

The government maximizes its growth rate subject to competitive decentralized equilibrium (CDE).

$$Max\ g = \phi\tau Z^{\alpha+\epsilon} - \delta_H, \tag{3.1}$$

Subject to the constraint:

$$\begin{aligned} &\frac{\alpha(1-\tau)Z^{\alpha-1}}{1-\nu(1-\sigma)} - \phi\tau Z^{\alpha+\epsilon}[1-\frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}] \\ &+\delta_h[1-\frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}] - \frac{\delta_k}{1-\nu(1-\sigma)} \\ &-\frac{m+b[(1-\tau)Z^{\alpha-1}-\phi\tau Z^{\alpha+\epsilon}+\delta_h-\delta_k]z}{1-\nu(1-\sigma)} = 0. \end{aligned} \tag{3.2}$$

So the static optimization lagrange comes out to be:

$$\begin{aligned} L = \phi\tau Z^{\alpha+\epsilon} - \delta_H + \lambda\Bigg[&\frac{\alpha(1-\tau)Z^{\alpha-1}}{1-\nu(1-\sigma)} \\ &-\phi\tau Z^{\alpha+\epsilon}(1-\frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}) \\ &+\delta_h\big(1-\frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}\big) - \frac{\delta_k}{1-\nu(1-\sigma)} \\ &-\frac{m+b\Big((1-\tau)Z^{\alpha-1}-\phi\tau Z^{\alpha+\epsilon}+\delta_h-\delta_k\Big)z}{1-\nu(1-\sigma)}\Bigg]. \end{aligned} \tag{3.3}$$

After taking the first order conditions (SEE APPENDIX **B**), the growth optimizing τ comes out to be:

$$\tau = \frac{-\alpha(-bz+\alpha-1)z^{\alpha-1}+bz(\delta_h-\delta_k)}{((\alpha-bz)\epsilon+\alpha)z^{\alpha-1}+b\phi z^{\alpha+\epsilon}}. \tag{3.4}$$

3.2 Numerical simulations for fiscal implications

This section uses simulations to explain the role of the government's growth maximizing fiscal policy. This is done by changing values of parameter involved in the model and the impact on τ and other important variables are observed that helps to maximize growth in the country.

3.2.1 Growth maximizing fiscal policy under multiple equilibria

This section will look at the changes in the Growth Maximizing (GM) fiscal policy because of changes in γ, σ, and b.

The table (3.1) look at the changes in the GM fiscal policy of the government. The first row shows the values under the benchmark parameters.

Table 3.1: Effect of change in parameters

	g	$\hat{\tau}$	Z^*	X^*	C/H	$\hat{\rho}$
$Benchmark Values$ $\gamma = 3.5, b = 0.1, \epsilon = 0.1$	0.0710	0.1079	20.92	1.3139	0.0627	0.1363
$\gamma = 1.5, b = 0.1, \epsilon = 0.1$	0.1018	0.1897	13.56	0.7826	0.0577	0.0832
$\gamma = 3.5, b = 0.2, \epsilon = 0.1$	0.0971	0.1904	12.28	0.8916	0.0725	0.1833
$\gamma = 3.5, b = 0.1, \epsilon = 0.05$	0.0698	0.1233	20.99	1.2870	0.0613	0.1337

Table 3.1 shows the result for numerical simulation when gamma decreases from 3.5 to 1.5 . Government increases its tax rate from 10.79% to 18.97%. This higher tax collected from agents is invested in human capital that helps to make individuals more patient. The rate of time preference is decreased from 0.13 to 0.08, also the consumption to human capital ratio decreases from

1.131 to 0.782 . Private investment to GDP ratio is decreased from 20.92 to 13.56. This is due to the fact that increased taxation by the government levies a burden on agents due to which they are left with less income as compared to the case where there is less tax. The tax revenue collected by the government is reinvested in human capital that leads the economy to a high growth path.

This could be illustrated by taking examples of two countries: the first country is where individuals value human capital more (γ=3.5), the second country is where agents give less value to human capital (γ=1.5). In the country where agents do not value human capital more, the government imposes a higher tax. This revenue collected by imposing higher taxes is invested in human capital in the economy that leads to an increase in growth rate. Also Z*, X* and C/H decrease in developing country as higher taxes lead to a decrease in the consumption for this country.

In our model b is the slope of impatience function. As b is increasing from 0.1 to 0.2, agents are becoming impatient. They put more weight on current consumption rather than future consumption. We know that the government seeks to maximize the growth in the country. For this scenario, the government is imposing higher taxes that would lead to a high growth rate in the economy. In the case of a developing country where agents are impatient, the government imposes higher taxes in order to maximize growth. Since our model is focused on human capital, this leads to growth in the economy because the government is spending more on human capital by imposing taxes on individuals. Also Z*, X* decrease due to the higher tax rate. The rate of time preference is high as agents are impatient.

ϵ is representing the degree of externality. By looking at the results of numerical simulations we can see that as the degree of externality is increasing from 0.05 to 0.1, the growth rate of the economy is increasing even with a smaller government size. The tax rate is decreasing because of the presence of a higher degree of externality in the economy.

We can compare two countries, one with lower externalities of human capital and a lower growth rate, and the second country with higher externalities and higher growth rate. In the first case, with low ϵ there is a higher tax rate as the government seeks to maximize growth. Also, X* and C/H are higher because consumption is more.

3.3 Conclusion

This chapter shows the result of optimal revenues collected by the government that will maximize the growth of the economy. It is clear from the results that optimal government revenues are dependent on endogenous time preferences. The positive externalities of human capital helps to reduce the size of government tax revenues and facilitate growth.

Table 3.2: Effect of change in parameters

	g^{GM}	$\hat{\tau}$	Z^*	X^*	C/H	$\hat{\rho}$
$\gamma \downarrow$	(+)	(+)	(−)	(−)	(−)	(−)
$b \uparrow$	(+)	(+)	(−)	(−)	(+)	(+)
$\epsilon \uparrow$	(+)	(−)	(−)	(+)	(+)	(+)

Table (3.2) shows the summary of numerical simulation results for growth maximizing fiscal policy. A closer look at γ shows that when agents value human capital less then the government steps in through its tax policies to achieve higher growth in the economy. This increased tax is then reinvested in human capital in the economy. Similarly for the case of the slope of the impatience function (b), when individuals in society are impatient, they value current consumption more than future than the government increases taxes to achieve the growth in the country. For a higher externality effect the size of the government is decreasing since there are already positive spillover effects of human capital that help the economy grow.

Conclusions

This thesis looked at the implications of human capital externalities incorporating the endogeneity of time preferences. This is done theoretically and numerically. The model is solved along the balanced growth path and long-run dynamics are analyzed. The solution of the model gives unique and multiple equilibria. Numerical simulation analysis is performed to look at how changing the parameters (σ, b, γ and ϵ) results in multiple equilibria. Patient individuals are willing to invest more in human capital and there exist more external effects. This overall increases the human capital accumulation in an economy.

Moreover, this thesis also determines growth maximizing fiscal policy. The government sets a tax rate that is used to maximize the welfare of the state and to push the economy to a high growth path. We analyze this through numerical simulation analysis by looking at different cases. The main findings suggest that an economy where individuals value human capital less would have to bear more taxes since the government would play its role by increasing tax to reinvest in human capital to increase growth. A similar pattern is observed for an impatient society where individuals would be taxed more. Externalities help the economy get out of a poverty trap and increases its growth rate. In such a country the role of the government would be smaller.

We can apply the result of our model in a developing country case where individuals do not know the importance of human capital. It is obvious that for the developing country case, a government has limited resources and we need to look at the best optimal policies for growth. Our results suggest that developing countries like Pakistan can get out of low growth traps the government follows policies promoting optimal human capital. This type of policy was also observed for the case of East Asian countries in the pre-1980 era that helped them to grow at a faster rate.

Bibliography

Agénor, P.R. (2010). A theory of infrastructure-led development. Journal of Economic Dynamics and Control, 932-950

Antoci, A., Galeotti, M., & Russu, P. (2011). Poverty trap and global indeterminacy in a growth model with open-access natural resources. Journal of Economic Theory, 146(2), 569-591.

Audretsch, D. B., Hlsbeck, M., & Lehmann, E. E. (2012). Regional competitiveness, university spillovers, and entrepreneurial activity. Small Business Economics, 39(3), 587-601.

Barro, R. J. (1990). Government spending in a simple model of endogenous growth. Journal of Polictical Economy, 103-125

Barro, R. J. (2001). Human capital and growth. The American Economic Review, 91(2), 12-17.

Bassanini, A., & Scarpetta, S. (2002). The driving forces of economic growth. OECD Economic studies, 2001(2), 9-56.

Becker, G. S., & Mulligan, C. B. (1997). The endogenous determination of time preference. The Quarterly Journal of Economics, 112(3), 729-758.

Blanchard, O. J. (1985). Debt, deficits, and finite horizons. Journal of political economy, 93(2), 223-247.

Bohm-Bawerk, Eugen V., The Positive Theory of Capital, William Smart, trans. (Freeport, NY: Books for Libraries Press, 1971 [1891]).

Burton A. Weisbrod (1962).Education and Investment in Human Capital. Journal of Political Economy, 70(5).

Carboni, O. A., & Russu, P. (2013). Linear production function, externalities and indeterminacy in a capital-resource growth model. Journal of Mathematical Economics, 49(5), 422-428.

Chaudhry, A., Tanveer, H., and Naz, R. (2017). Unique and multiple equilibria in a macroeconomic model with environmental quality: an analysis of local stability. Economic Modelling, 206-214.

Choi, H., Mark, N. C., & Sul, D. (2008). Endogenous discounting, the world saving glut and the US current account. Journal of International Economics, 75(1), 30-53.

Dinda, S. (2008). Social capital in the creation of human capital and economic growth: A productive consumption approach. The Journal of Socio-Economics, 37(5), 2020-2033.

Dioikitopoulos, E. V., & Kalyvitis, S. (2015). Optimal fiscal policy with endogenous time preference. Journal of Public Economic Theory, 17(6), 848-873.

Dioikitopoulos, E. V., & Kalyvitis, S. (2010). Endogenous time preference and public policy: Growth and fiscal implications. Macroeconomic Dynamics, 14(S2), 243-257.

Epstein, L. G., & Hynes, J. A. (1983). The rate of time preference and dynamic economic analysis. Journal of Political Economy, 91(4), 611-635.

Fan, Q., Goetz, S. J., & Liang, J. (2016). The interactive effects of human capital and quality of life on economic growth. Applied Economics, 48(53), 5186-5200.. The interactive effects of human capital and quality of life on economic growth. Applied Economics, 48(53), 5186-5200.

Fernndez, E., Prez, R., & Ruiz, J. (2012). The environmental Kuznets curve and equilibrium indeterminacy. Journal of Economic Dynamics and Control, 36(11), 1700-1717.

Fisher, Irving, The Theory of Interest (New York: Macmillan, 1930).

Fuchs, V. R. (1982) Time preference and health: An exploratory study, in Economic Aspects of Health, V. R. Fuchs, ed., pp. 93-120. Chicago: University of Chicago Press.

Galor, O., & Zeira, J. (1993). Income distribution and macroeconomics. The review of economic studies, 60(1), 35-52.

Gaspar, J., Vasconcelos, P. B., & Afonso, O. (2014). Economic growth and multiple equilibria: A critical note. Economic Modelling, 36, 157-160.)

Harrison, G.W.,M. I. LAU, & M. B. Williams (2002) Estimating individual discount rates in Denmark: A field experiment, American Economic Review 92, 1606-1617.

Hausman, J. A. (1979). Individual discount rates and the purchase and utilization of energy-using durables. The Bell Journal of Economics, 33-54.

Hosoya, K. (2012). Growth and multiple equilibria: A unique local dynamics. Economic Modelling, 1662-1665.

Horowitz, J. K. (1991). Discounting money payoffs: An experimental analysis. Handbook of behavioral economics, 2, 309-324.

Hosoya, K. (2016). Recovery from natural disaster: A numerical investigation based on the convergence approach. Economic Modelling, 55, 410-420.

Hosoya, K. (2014). Public health infrastructure and growth: Ways to improve the inferior equilibrium under multiple equilibria. Research in Economics, 68(3), 194-207.

Itaya, J. I. (2008). Can environmental taxation stimulate growth? The role of indeterminacy in endogenous growth models with environmental externalities. Journal of Economic Dynamics and Control, 32(4), 1156-1180.

Kneller, R., Bleaney, M. F., & Gemmell, N. (1999). Fiscal policy and growth: evidence from OECD countries. Journal of Public Economics, 171-190.

Kuo, c.-c., & Yang, c.-h. (2008). Knowledge capital and spillover on regional economic growth: Evidence from China. China Economic Review, 594-604.

Ladrn-de-Guevara, A., Ortigueira, S., & Santos, M. S. (1997). Equilibrium dynamics in two-sector models of endogenous growth. Journal of economic dynamics and control, 21(1), 115-143.

Lawrance, E. C. (1991). Poverty and the rate of time preference: evidence from panel data. Journal of Political economy, 99(1), 54-77.

Lucas, R. E., & Stokey, N. L. (1984). Optimal growth with many consumers. Journal of Economic Theory, 32(1), 139-171.

Lucas, R, (1988), On the mechanics of economic development, Journal of Monetary Economics 22, pp. 3-42.

Meier, S., & C. D. Sprenger (2010) Stability of time preferences. Discussion Paper No. 4756, IZA.

Nelson, R. R., and Phelps, E. S. (1966). Investment in humans, technological diffusion, and economic growth. The American economic review, 56(1/2), 69-75.

Ortigueira, S., & Santos, M. S. (1997). On the speed of convergence in endogenous growth models. The American Economic Review, 383-399.

Palivos, T., Wang, P., & Zhang, J. (1997). On the existence of balanced growth equilibrium. International Economic Review, 205-224.

Pelinescu, E. (2015). The impact of human capital on economic growth. Procedia Economics and Finance,22, 184-190.

Pender, J. L. (1996). Discount rates and credit markets: Theory and evidence from rural India. Journal of development Economics, 50(2), 257-296.

Perez-Arce, F. (2011) The effects of education on time preferences. Working Paper 844, RAND.

Romer, P. M. (1989)." Human Capital and Growth: Theory and Evidence..NBER Working Paper, (3173).

Samuelson, P. A. (1937). A note on measurement of utility. The Review of Economic Studies, 4(2), 155-161.

Samwick, A. A. (1998). Discount rate heterogeneity and social security reform. Journal of Development Economics, 57(1), 117-146.

Schmitt-Grohé, S., and Uribe, M. (2003). Closing small open economy models. Journal of international Economics, 61(1), 163-185.

Schumacher, I. (2009). Endogenous discounting via wealth, twin-peaks and the role of technology. Economics letters, 103(2), 78-80.

Strulik, H. (2012). Patience and prosperity. Journal of Economic Theory, 147(1), 336-352.

Turnovsky, S. J. (2000). Methods of macroeconomic dynamics. MIT press.

Uzawa, H. (1968). Time preference, the consumption function, and optimum asset holdings. Value, capital and growth: papers in honor of Sir John Hicks. The University of Edinburgh Press, Edinburgh, 485-504.

Xie, D. (1994). Divergence in economic performance: transitional dynamics with multiple equilibria. Journal of Economic Theory, 63(1), 97-112.

Appendix A

The current value Hamiltonian function for the model is

$$H = \frac{(C^{\nu} H^{\gamma})^{1-\sigma} - 1}{1-\sigma} e^{-\Delta(t)} + \lambda[(1-\tau)K^{\alpha}H^{1-\alpha} - C - \delta_k K] \tag{A-1}$$

The necessary first order conditions are:

$$\frac{\partial H}{\partial C} = \nu C^{\nu(1-\sigma)-1} H^{\gamma(1-\sigma)} e^{-\Delta(t)} - \lambda = 0 \tag{A-2}$$

By rewriting the above expression we get:

$$\lambda = \nu C^{\nu(1-\sigma)-1} H^{\gamma(1-\sigma)} e^{-\Delta(t)} \tag{A-3}$$

The above equation shows the expression for the shadow price. -

$$\dot{K} = (1-\tau)K^{\alpha}H^{1-\alpha} - C - \delta_k K, \ , 0 < \tau < 1, \tag{A-4}$$

$$\dot{\lambda} = -\partial H/\partial K = \dot{\lambda} = -\lambda\alpha(1-\tau)K^{\alpha-1}H^{1-\alpha} + \delta_k \lambda \tag{A-5}$$

Taking ln of expression (A-3) and differentiating with respect to time t, the growth rate of consumption is given as:

$$\frac{\dot{C}}{C} = \frac{1}{\nu(1-\sigma)-1}\left[\frac{\dot{\lambda}}{\lambda} - \gamma(\frac{\dot{H}}{H}) + \dot{\Delta}(t)\right] \tag{A-6}$$

Replacing value of $\frac{\dot{\lambda}}{\lambda}$ and $\frac{\dot{H}}{H}$ in above expression, we get

$$\frac{\dot{C}}{C} = \frac{1}{\nu(1-\sigma)-1}[(-\alpha(1-\tau)(\frac{K}{H})^{\alpha-1}$$
$$+\delta_k\lambda - \gamma(1-\sigma)\tau\phi(\frac{K}{H})^{\alpha+\epsilon} + \gamma(1-\sigma)\delta_h + \rho] \tag{A-7}$$

We are defining two new variables to solve our model as $X = C/K$, and $Z = K/H$

$$\frac{\dot{X}}{X} = \frac{\dot{C}}{C} - \frac{\dot{K}}{K} \tag{A-8}$$

Substituting values of $\frac{\dot{C}}{C}$ and $\frac{\dot{K}}{K}$ in above expression, we get

$$\begin{aligned}\frac{\dot{X}}{X} = \frac{1}{\nu(1-\sigma)-1}[(-\alpha(1-\tau)(\frac{K}{H})^{\alpha-1} + \delta_k \\ -\gamma(1-\sigma)\tau\phi(\frac{K}{H})^{\alpha+\epsilon} + \gamma(1-\sigma)\delta_h] - \\ (1-\tau)K^{\alpha}H^{1-\alpha} - \frac{C}{K} - \delta_k\end{aligned} \tag{A-9}$$

$$\begin{aligned}\frac{\dot{X}}{X} = \frac{1}{\nu(1-\sigma)-1}[(-\alpha(1-\tau)Z^{\alpha-1} \\ -\gamma(1-\sigma)\tau\phi Z^{\alpha+\epsilon} + \gamma(1-\sigma)\delta_h + \rho(.) + \delta_k] - \\ (1-\tau)Z^{\alpha-1} + X + \delta_k\end{aligned} \tag{A-10}$$

$$\frac{\dot{Z}}{Z} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H} \tag{A-11}$$

Substituting values of $\frac{\dot{K}}{K}$ and $\frac{\dot{H}}{H}$ in above expression, we get:

$$\frac{\dot{Z}}{Z} = (1-\tau)(\frac{K}{H})^{\alpha-1} - \frac{C}{K} - \delta_k - \tau\phi\left(\frac{K}{H}\right)^{\alpha+\epsilon} + \delta_h \tag{A-12}$$

$$\frac{\dot{Z}}{Z} = (1-\tau)Z^{\alpha-1} - X - \tau\phi Z^{\alpha+\epsilon} + \delta_h - \delta_K \tag{A-13}$$

Appendix B

The lagrangian for growth maximizing fiscal policy is:

$$
\begin{aligned}
L = \phi\tau Z^{\alpha+\epsilon} - \delta_H + \lambda \frac{\alpha(1-\tau)Z^{\alpha-1}}{1-\nu(1-\sigma)} \\
-\phi\tau Z^{\alpha+\epsilon}[1 - \frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}] \\
+\delta_h[1 - \frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}] - \frac{\delta_k}{1-\nu(1-\sigma)} \\
-\frac{m + b[(1-\tau)Z^{\alpha-1} - \phi\tau Z^{\alpha+\epsilon} + \delta_h - \delta_k]z}{1-\nu(1-\sigma)}
\end{aligned}
\tag{B-1}
$$

The necessary first order conditions with respect to z and τ are given as:

$$
\begin{aligned}
\frac{\partial L}{\partial z} = \frac{\phi\tau z^{\alpha-\epsilon}(\alpha-\epsilon)}{z} + \lambda\Bigg[\frac{\alpha(1-\tau)z^{\alpha-1}(\alpha-1)}{z(1-\nu(1-\sigma))} \\
-\frac{\phi\tau z^{\alpha+\epsilon}(\alpha+\epsilon)[1 - \frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}]}{z} \\
-\frac{b[\frac{(1-\tau)z^{\alpha-1}(\alpha-1)}{z}\frac{\phi\tau z^{\alpha+\epsilon}(\alpha+\epsilon)}{z}]z + b((1-\tau)z^{\alpha-1} - \tau\phi z\alpha + \epsilon + \delta_h - \delta_k)}{1-\nu(1-\sigma)}\Bigg]
\end{aligned}
\tag{B-2}
$$

$$
\frac{\partial L}{\partial \tau} = \phi z^{\alpha+\epsilon} + \lambda[-\frac{\alpha z^{\alpha-1}}{1-\nu(1-\sigma)} - \phi z^{\alpha+\epsilon}[1 - \frac{\gamma(1-\sigma)}{1-\nu(1-\sigma)}] - \frac{(b - z^{\alpha-1} - \phi z^{\alpha+\epsilon})z}{1-\nu(1-\sigma)}]
\tag{B-3}
$$

Solving 3.19 for λ we get,

$$
\lambda = \frac{\phi z^{\alpha+\epsilon}(\nu\sigma - \nu + 1)}{z^{\alpha+\epsilon}\gamma\phi\sigma - z^{\alpha+\epsilon}b\phi z + z^{\alpha+\epsilon}\nu\phi\sigma - z^{\alpha+\epsilon}\gamma\phi - z^{\alpha+\epsilon}\nu\phi - z^{\alpha-1}bz + \phi z^{\alpha+\epsilon} + z^{\alpha-1}\alpha}
\tag{B-4}
$$

$$
\frac{\partial L}{\partial \lambda} = \frac{\phi z^{-\epsilon+1}(\nu\sigma - \nu + 1)}{-z^{-\epsilon+1}\gamma\phi\sigma - z^{-\epsilon+1}\nu\phi\sigma + z^{-\epsilon+2}b\phi + z^{-\epsilon+1}\gamma\phi + z^{-\epsilon+1}\nu\phi - z^{-\epsilon+1}\phi + bz - \alpha}
\tag{B-5}
$$

After putting value of λ in 3.18 and solving for τ, the simplified expression for τ is:

$$\tau = \frac{-\alpha(-bz + \alpha - 1)z^{\alpha-1} + bz(\delta_h - \delta_k)}{((bz - \alpha)\epsilon + \alpha)z^{\alpha-1}b\phi z} \tag{B-6}$$