# Child Nutrition, Education & Child Labor: Impact on Human Capital Accumulation

by

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# Contents

# 1 Abstract

The study using a two-period overlapping generations model analyzes the complementing nexus between child nutrition and child education and how it alters parent's fertility decisions. The results of the model show that in the intermediate phase, the economy experiences a demographic transition. In this interval the child quantity-quality trade-off is observed but at a later stage the continuing process of increasing human capital allows agents to generate adequate resources to rear more children and simultaneously endowing them with the capacity of providing the children with education and nutrition. Minimum level of fertility in the model is attained when maximum child nutrition and time devoted towards education are attained. When human capital exceeds the maximum threshold level time devoted to education becomes a constant i.e. further increases in human capital has no effect on the time devoted to education pertaining that maximum capacity to learn has been reached. The analysis yields three steady-state level of equilibrium. Parents having low human capital prefer child quantity over child quality and hence are trapped in the low steady-state where both child labor as well as an undernourished population is dominant. The medium steady-state is considered to be desireable for the economy in our model since it is characterized by high human capital, low fertility and high child nutrition. Moreover, the model also provides an insight on child labor and its implications for the economy.

## 2 Introduction

In the field of economics, whenever economists contemplate about "human capital", they often refer it to education. A vast body of literature documents investment in education playing a major role not only in increasing future returns in the form of higher earnings but also in explaining the variations in wages and incomes of adults which has implications both at the household level as well as the country level. However, this poses a question in mind that what influences a child's educational success? Is it family background and circumstance? Is it adult income or simply child's characteristics such as aptitude, motivation, ability or behavior? While many things in 'background' might matter, recently research associates health as a potent factor, more specifically childhood health. Given that education is a main element for human capital accumulation, determining how childhood health either directly or indirectly augments human capital accumulation via academic performance and how it acts as a mechanism for intergenerational transmission is an essential question.

Taking into account the fundamental role played by health and education in the intergenerational transmission process, this paper aims at studying the impact of child nutrition and education on human capital accumulation. It focuses on the interplay between three factors: trade-off between health (measured through child nutrition) and education, quantity-quality trade- off of children and how inherited human capital captures the intergenerational effects. This research is designed to consider competition of resources between health and education, how an increase in the allocation of resources in one reduces the resources available for the other and thus accordingly alters parents fertility decisions. Moreover, the research also analyzes how these factors impede the productivity and efficiency of human capital of the future generations. Thus, the objective of the paper is two-fold. Firstly, the paper analyses the complementing nature and nexus between child nutrition and education. Secondly, it studies its implications on human capital accumulation.

A large part of the work done in this field is empirical in nature. The theoretical work done revolves around different channels and transmission mechanisms. Initially, literature explored the divergences and discrepancies in income and growth in countries. Why are some countries rich while others are poor? Why have some countries been able to successfully take off while others still remain stuck in their initial conditions? These questions formed the basic foundation of research by many economists, in order to determine the underlying factors responsible for such discrepancies and which up to date continues to puzzle them.

For the last half a century, Solow attributed such disparities as being brought about by differences in the accumulation of physical capital per worker however, since the mid 80's, research in economic growth accelerated with the advent of the "new growth theories" which mainly directed their attention towards productivity advances as a result of technological progress and increased human capital in the form of education.

The influence that education has on economic growth had been widely acknowledged both empirically and theoretically for decades now. Lucas (1988) initially emphasized the importance of human capital formation in enhancing growth however, the model only considered education as a sole vital factor for growth. It overlooked a key aspect, that human capital formation was dependent on how able bodied the people were. In short, human capital services can only be provided effectively and efficiently if people are healthy, thus warranting a closer look in such an area to observe how variations in health could influence growth and development via education. It was only until recently that health and education gained widespread recognition in boosting the process of economic development.

In truth Mankiw, Romer and Weil (1992), were the first to lay emphasis on a broader concept of human capital which incorporated the concept of education with health and nutrition. They contradicted Solow's model by showing that when adding human capital to it, the model best explained convergences and cross country differences in per capita income. In contrast, the original Solow model rather than explaining convergences, predicted that countries reached different steady states. However, it was only several years later that the interaction between economic growth and health gained widespread recognition and became the focal point of economic debate among economists. As a matter of fact Fogel (1994), Barro and Sala (1995) and Barro (1996) were pioneers in analyzing the liaison between economic growth and health which led to a substantial rise in the works of child health and education.

With respect to the trade off between health and education, the literature emphasized on both the complementing as well as the substitution effects of both factors. Zon & Muysken (2000) and Fuchs (1982) emphasized on health and education acting as complements by reinforcing each other. Zon & Muysken (2000) supported such a notion by stating that a reduction in the health of the individuals minimized the total number of working days. Hence, through such a perspective both were complements as deterioration in health lowered the total supply of human capital. Similarly Fuchs (1982) augmented such a notion by arguing that an increase in the investment of health capital increased an individuals life expenctancy allowing it to invest more in human capital to reap greater benefits in the future.

In contrast, Grossman (1972) was the first to come up with the proposition that health and education could act as substitutes by competing against each other for resources. He developed the notion that resources were a prerequisite for providing health. As a result a direct trade-off existed between health and education whereby an increase in the allocation of resources towards the health sector could stimulate growth via an improvement in the health of the masses however, he argued that growth could also be brought about by reducing the resources allocated to the health sector thus enabling the economy to use the freed resources for increasing human capital accumulation which could consequently act as an alternative way to promote growth.

Numerous factors are influenced by health which promotes growth. Foremost, variations in health influences income. Such a notion has been widely accepted, recognized and tested empirically and theoretically. Weil (2007) used historical data to estimate the impact of health on GDP and found that reductions in health variances lowered the variance of GDP per worker by 9.9%. Reducing such gaps also lowered ratio of income of the country at the 90th percentile to the 10th percentile. Arora (2001) established in a time series study consisting of 62 health related variables that 26% to 40% of the growth is endorsed by them. Devlin and Hansen (2001) established Granger causality between health and GDP for OECD countries and similar findings have been corroborated by various economists in their studies, the most prominent being Barro and Sala-i-Martin (1995). Barro (1991), Easterly and Levine (1997), Gallup and Sachs (2000) worked along the same lines but used life expectancy or mortality as a measure of health and found a positive and significant impact. Bloom, Canning and Sevilla (2001) as opposed to the standard human capital models included both health and work experience and found both to have a positive impact on aggregate output. However, at higher levels of income, impact of health on growth is either insignificant or even negative (Van-Zon and Muysken, 2001).

Good health is a vital factor for the welfare and prosperity of an individual. An improvement in health induces a higher human capital level, increased economic productivity and economic growth of a country. It increases labor productivity through enhancement of an individual's physical capabilities, strength and endurance, allowing them to produce more in a given time period by reducing the number of days lost in sickness. Furthermore, it has a direct impact on cognitive ability, learning and academic performance.

This impact influences in two ways. First, better child health leads to higher school attendance by lowering absenteeism rates thereby causing an increase in the cognitive ability of the child as is evident from numerous empirical studies and Randomized Control Trials (RCT's). For instance Miguel & Kremer (2004) found that deworming reduced absenteeism rates in schools and increased school participation in Kenya. Similarly Bleakly (2003) found that deworming led to higher school attendance and increase in literacy in the American South. Bleakly, Costa, & Muney (2013) using data of United States for cohorts between 1820 and 1990 observed high correlations between health and education where height was taken to be a measure of health. OLS regression results showed that an increase in height by 1.2 centimeters was associated with an increase in schooling by 0.1 years suggesting investment in health and education as valuable for development.

The second mechanism involves health lowering mortality thereby increasing the life expectancy of an individual and consequently its decision to invest more in education in order to reap greater benefits from it in the future. This effect has been analyzed both empirically as well as theoretically. For example, Behrman & Deolalikar (1988) shows that if children take on the responsibility of caring for parents and when higher returns are expected from male labor, then efficient decision for adults is to prefer boys. Similarly, if health has an impact on schooling returns, then it would be more efficient to invest in healthy students.

Kalemli-Ozcan, Ryder, & Weil (2000) construct a continuous time overlapping generation's model to show how the probability of an increase in life expectancy raises optimal schooling investment decisions in order to earn higher returns over a longer time horizon. Further calibration of the model yielded statistically significant results between education and life expectancy with 1% increase in life expectancy resulting in raising schooling by seven-tenths of one percent. Osang & Sarkar (2008) also construct a similar OLG model however it incorporates lifetime uncertainty to show its affect in lowering both physical and human capital which can be counteracted only if government undertakes high expenditure on education and health thereby introducing the concept of private and public spending on both factors. Ricci & Zachariadis (2007) work along the same lines by building a theoretical model to assess the relationship between life expectancy, education, public and private spending on health and income and concluding these factors to be positively related with tertiary education as being vital for longevity rather than basic education.

Psacharopoulos (2006) and Grossman (1999) also lay emphasis on health being a factor in enhancing education. Grossman (1999) initially constructed a model for the demand for health and in particular stated that health and human capital were two different entities, the former affecting an individual's time spent in earnings and producing goods while knowledge he stated only had an impact on its market and non-market productivity. Psacharopoulos (2006) combined macro and micro level evidence to show how education entails sacrificing present earnings to gain higher returns in the future and highlights the social costs incurred by a state through inadequate education.

These aforementioned models even though being useful for economic analysis have their limitations. Foremost, they overlook the causation between life expectancy and education. On the one hand, the probability of an increase in life expectancy increases the returns to investing in education whereas more of education leads to a further improvement in life expectancy through three channels (Ricci & Zachariadis, 2007). First, an educated individual earns higher income in the future enabling it to purchase quality health care thereby increasing its chances of longevity. Second, more affluent individuals are able to provide more of quality basic education to their children which increases the efficiency of the child's investment in health. Lastly, an improvement in education especially among the labor force enhances the efficiency of their health investment which further increases their chances of longevity.

Moreover, all these models claim that health improvements result in economic growth. Literature on the other hand shows such a relationship to be murky and vague. Such a widely popular belief was contradicted initially by Acemoglu and Johnson (2007). Using data between 1940 and 1980, and regressing income on life expectancy reported a negative effect. Similarly, Ashraf, Lester & Weil (2008) show that health improvements only lead to modest increases in GDP per capita and that too with considerable time lags. Their simulated model using data on schooling, labor productivity, demographics, disease and national income depicted that an increase in life expectancy at birth from 40 to 60 led to a 15% increase in GDP per capita in the long run. Further, removing malaria and tuberculosis in Sub-Saharan Africa, results in GDP rising by 2% in the long run. Their analysis yields that even though an improvement in health positively influences worker productivity, it generates a negative effect in the form of a growing population and therefore, suggest policies involving family planning, foreign investment to reduce capital dilution and planning beforehand for extra teachers and schooling facilities to accommodate a larger population. They emphasize the benefits from long run gains on income from health and that decline in mortality leads to fertility adjustments over a span of 50 years. In contrast, Lorentzen et.al (2008) shows a positive effect of life expectancy on GDP.

In response to both these contradictory findings Aghion et.al (2011) emphasized the inclusion of health and health accumulation. By jointly looking at both these factors, they ran cross-country regressions from 1960-2000 using an Instrumental Variables approach where they instrument Malaria Ecology Index and geographic and climatic variables and found a positive impact of both variables on GDP per capita.

Type of healthcare, whether public or private also matters. Shi & Dzhumashev (2015) theoretically and empirically test the significance of child health investment. In their theoretical analysis, they observe three types of healthcare systems: public, private and hybrid. Using an OLG model, the possibility of how child health investment impacts human capital accumulation as well as fertility is observed. Under the hybrid system, health investment increases education investment whereas fertility declines. Empirical econometric estimation for Australian children further shows a significant impact of health on children's education. Bhattachary & Qiao (2005), and Agenor (2009) also analyze the implication of public or private healthcare.

Better and improved health also generates a knock-on effect in that it allows resources to be allocated for alternative uses which would otherwise have been used for curative or preventive healthcare measures. It therefore impacts savings. The probability of an increase in life expectancy acts as an incentive for the adult population to save for retirement. Additionally, these savings could be used productively by the healthy population for investment purposes which could generate income translating into growth. In contrast, an individual with ill-health would find itself mired in poverty due to lower savings mainly as a result of higher health care expenditures which consequently reduces business investment. Chakraborty (2004) focuses around such an aspect and examines how a longer life expectancy and lower mortality facilitates growth and development. He constructs a two-period overlapping generations model and incorporates mortality risk in it. Survival probability is endogenously determined by public investment in health. The results show that in low-income countries life expectancy is short; postponement of consumption is lower hence leading to low savings and investment. Human capital investment is hence lower due to lower expected returns. This implies that mortality risk could act as a deterrent for development. Bloom & Canning (2005) focus on mortality but differ in that they see the impact of adult survival rate on labor productivity and found a positive association between both. Aísa & Pueyo (2004), Chakraborty & Das (2005), Castelló-Climent & Doménech (2008) too lay emphasis on health investments acting as a stimulant to reduce mortality and consequently improve growth.

Recent papers stress the importance of varied mechanisms influencing life expectancy foremost being disease and infection. In this regard Huang et. al (2010) theoretically and empirically shows how HIV/AIDS impacts life expectancy and consequently human capital and income growth. Theoretically, using a three-period OLG model and incorporating probability of premature death in all life stages it is shown that such a variable decreases human capital and slows growth. Empirically, the results attest such an impact.

Moreover, previous research is more concerned with how income effects health or vice versa, how health effects labor productivity, academic performance, how different health systems, quality of medical care received, infectious diseases, health information outreach are conducive for growth and how such effects due to various environmental problems translate negatively well into the future. It overlooks a vital aspect necessary to improve health i.e. nutrition or caloric intake.

Inadequate nutrition causes many hindrances especially in achieving child quality. Hunger leads to inattentiveness. Protein-energy malnutrition is associated with lower cognitive ability especially in areas of problem solving, critical thinking and capacity. Micronutrient-deficiency disorder is a further hindrance in academic performance. Iodine deficiency lowers intelligence and causes pisco-motor retardation and cretinism. Iron deficiency anemia, common among children, leads to a reduction in mental and motor development test scores. Vitamin A deficiency creates eyesight problems. Helminthic infection produces morbidity which further impairs cognitive abilities and cosequently higher absenteeism and attrition.

Malnutrition further aggravates a child's condition is reinforced by Larrea, Freire and Lutter (1998). In their paper they show that malnutrition leads to stunting and is established in the first 2-3 years of a child's life. Such cases are prominent where supplementation programs are not provided, thereby augmenting the importance of such programs. Additionally, some randomized experiments also stressed on nutritious food and supplementation in childhood for successful development in the future. One such a study conducted in Jamaica provided nutritional supplements to growth-stunted children and found that it significantly increased their test scores but led to deterioration in their scores once the supplementation ended (Walker, Chang, Powell, & Grantham, 2005).

A large literature exists which supports the notion that nutrition is important for growth and development. Fogel found nutrition conducive for economic growth. Arcand (2001), found nutrition to be fundamental for economic growth leading to nutrition-poverty traps in the possibility of undernourishment and found both empirically and theoretically that nutrition matters for an increase in output per capita. Moreover, “Copenhagen Concensus”, includes in its list the top most developmental projects with high benefit to cost ratio. The first thirteen developmental projects are all health related, supporting the vitality of health. Swamy (1997), Vasilakis (2011), all emphasize nutrition as a key aspect in promoting growth and development.

A considerable body of research also emphasizes the intergenerational effects of health. Such a perspective emphasizes that since health is a stock thus, negative shocks early in life especially during periods of a child’s development could have implications for the individual in the future. This theory was initially introduced by Barker (1997), who stated that the development and health of an embryo depended on an adequate supply of nutrients as well as oxygen and if these necessary factors were in short supply it could hinder the development of important organs in the human body and make the fetal more susceptible to diseases such as heart disease, strokes, diabetes and hypertension later in life. Additionally, under-allocation of such resources results in undernourished babies being born with lower birth weights and lower height and these factors researchers have observed to have undermined their ability to attain education which further impedes their productivity and earnings in the labor market. Overall, Barker emphasized on nutrition of both the mother and the child as an integral factor for child quality.

Studies indicate that if at utero, these factors are not taken care of then it leads to intrauterine growth retardation (IUGR), a term used for those children who do not reach their actual growth. In this regard, mothers build, height, weight, nutritional adequacies are important both before conception as well as during pregnancy. Apart from nutrition, various other factors contribute towards IUGR such as intestinal infections as a result of parasites, improper hygiene, smoking, diarrheal disease as well as malaria (ACC/SCN, 2000). Such a condition is more often proxied through low birth weight.

For school age children, the extent to which nutrition depletion causes inadequate skeletal growth or results in insufficient accumulation of mass and fat is analyzed. In short, the tool anthropometry is often used to evaluate their nutritional status. Under such a measure height for age is particularly useful especially in estimating stunting attributes.

Poor anthropometric status is as a result of numerous factors including low birthweight. Many studies have shown a correlation between stature and low birthweight. Furthermore, it has been found that the first two years of a child are very crucial and important for its growth and development. Nutritional inadequacy in these two years could have adverse effects in these infants in the form of growth retardation. Such conditions are amplified if insufficient care is provided and aggravated during these years because children have high nutritional requirements and simultaneously are not able to voice their needs. Failure to identify their needs and provision of unhealthy foods could falter their growth and make them more prone to disease and infection (Chen 2006). Thus, the reason for growth retardation in developing countries.

Various studies have shown that inadequate growth in the first two years are most often either lost in the forthcoming years or only partially regained. Such conditions worsen if unfavorable environment is provided to the child (Martorell, et al. 1994). For instance, Hoddinott and Kinsey (2001), in their study found that children exposed to the 1994/95 drought in rural Zimbabwe had height for age z-scores well below as compared to children not exposed to the drought. However, it was noted that older children did not face such consequences corroborating the fact that child development has certain sensitive phases during which a negative stimuli or shock adversely effects its development and which in most cases are irreversible.

Some of the data on natural experiments have corroborated such an aspect. Ravelli, et al., (1998) conducted a study of individuals who as prenatals were exposed to the 1944-1945 famine conditions in Amsterdam and compared them with those born before and after the famine. The results revealed that individuals who were born during the famine period suffered from diabetes as adults. Similarly Chen & Zhou (2007) examined the effect of the 1959-1961 famine in China on the health of the individuals later in life. The OLS results of the natural experiment showed that individuals born during this period experienced adverse effects on their height, aggregarian income and labor supply. Currie & Almond (2010) provides a comprehensive review of various studies focusing on how early life environment affects the future. They conclude that events before the age of five are crucial for the development and well-being of a child and provide remediation strategies such as in-kind and cash transfers in overcoming damaging negative events in the child's life.

Several birth cohort studies using a variety of data sets have also empirically tested such a notion with statistically significant results. One such a study was conducted by Wadsworth & Kuh (1997) in which 5,362 respondents born in 1946 were followed till the ages of 43 and 51 and data on their physical and mental health and dietary habits were obtained by interviewing their mothers, nurses and physicians. Data on their cognitive ability was collected through school tests and their occupation, income and circumstances were assessed by interviewing them during their adult years. This 50 year study emphasized through its results that early childhood environment mattered the most as children exposed to poor conditions were found to be more susceptible to systolic blood pressure and expiratory diseases. Moreover, those experiencing walking and talking difficulties were more likely to suffer from schizophrenia by the age of 43.

Another compelling study by Case, Fertig, & Paxson (2003) quantified, using the 1958 National Child Development Study (NCDS) which followed individuals from birth to age 42 in Britain, how childhood health and economic circumstances had an adverse impact on an individual's cognitive ability, employment and earnings. The results portrayed by the study laid emphasis on maternal health as being important for satisfactory child health later in life as it found that smoking mothers had a higher likelihood of their children scoring less in their English O-Levels as well as suffering from Attention Deficit Hyperactivity Disorder. Network in the family exacerbated the situation as wives with smoking husbands had a higher probability of smoking during pregnancy than those with non-smoking husbands which critically affected fetal development. Additionally, the study reported that chronic conditions in childhood reduced the chances of an individual being employed in adulthood.

Other studies such as those of Case & Paxson (2010), Currie (2008), Almond, Currie, & Herrmann (2011) and Meara (2001) using different data sets (Whitehall II and HRS) and recording health of individuals through different measures such as self-reporting and Activities of Daily Living (ADL) observe how lower heights among children drastically effect their cognitive abilities and employment opportunities in the future. Moreover, mothers receiving no prenatal care were found to have babies with low birth weight and height.

Various theoretical studies using OLG models also emphasize on the implications of childhood health on adult health and their impacts on human capital accumulation, productivity and economic growth while some study how variations in health status could lead to income inequalities and poverty traps resulting in a state experiencing multiple equilibria (Agénor 2009; Wang & Wang 2013; Rivera, Currais, & Rungo 2006).

Contemporary research has built from such a concept, the idea of early child development (ECD). Such a theory has diversified notions those involving an amalgamation of physical, mental as well as social development. Such programs address issues related to nutrition, health, intelligence and social interaction of children. ECD programs encourage parents to participate along with their children. Children in such programs receive nutritional supplementation and health care while parents get guidance on how to best care for children. The significance of such programs for child development both physically and mentally is supported by extensive research by pediatrics, economists and sociologists.

Participants of such programs showed a remarkable improvement in children in terms of a higher IQ level as well as improvement in speech, hearing, listening and reading. Dropout rates and lower performances declined while the enthusiasm to progress towards higher levels amplified (Karoly et al 1998). Moreover, it was associated with a decline in cases of malnutrition, stunting, morbidity and mortality. Social adaptation and health care increased among such children and delinquency declined (Karoly et al 1998).

These economic effects of health are apparent both at the individual and the macroeconomic level. Despite, the literature being rich in its contribution towards health and its impact on the economy, many issues persist when accounting for such a factor. The real dispute and difficulty lies in assessing the magnitude of the health impact. Many issues persist when taking into account the impact of health on economic growth resulting in a wide research void. Foremost, is the lack of consensus among the economists as to how to measure health? Different studies use varied measures of health such as self-reporting, activities of daily living (ADL), medical records, recall ability, and anthropometric measures. Each of these measures may not completely capture the health status of the individual for example studies using self-reporting as a means to measure the health of the individual require them to rate their health from good health to poor health. However, the problem in such a measurement lies in the fact that what may be perceived by a certain individual as good health may in fact be considered as poor health by another. Such issues lead to measurement errors and thus question the validity of the study. Another concern is the issue of causality, more persistent in cases where health is associated with income/wealth making it difficult to disentangle the effects of both. Such an issue creates an endogeneity problem between both variables. While the ability to earn more allows an individual to increase its consumption of health related products like medicines, likewise good health facilitates higher earnings indirectly via improvement in the level of education and enhancement of one's participation in the labor market. This results in a feedback effect resulting in estimation problems when analyzing its effect on economic growth. Though the two-way relationship is difficult to disentangle, yet it is important in order to yield accurate and unbiased results. Third, is the issue of timing. In order to study the implications of health well into the future, results in huge time lags.

This complicates matters by making it difficult to assess the relationship of health with different variables and its impact on the economy at the macroeconomic level in the long run. Thus, such large research gaps and shortcomings allow room for further research in this area.

In account of the aforementioned shortcomings, our study amalgamates health, education and intergenerational effects in one theoretical model. The theoretical paper in line with our study is Rivera, Currais, & Rungo, (2006). However, our study differs from them in certain aspects. First, our study considers a private market for education as opposed to the base papers assumption of public market for education. Due to the assumption of a private market for education in our study, individuals have to pay for the education of their child. Since education is provided by teachers, all individuals incur a cost per child equivalent to the average human capital of the teacher which equals the average human capital of the population. In a homogeneous population the average human capital of the teacher is equivalent to the adult human capital. The model incorporates the cost of education in order to observe the complementarity between health and education i.e. when more/less consumption is being diverted towards education, how it effects the consumption of health.

Second, the human capital technology in the base paper considers only education and health as vital factors. However our study introduces inherited human capital along with health and education in the human capital accumulation function. This factor allows us to analyze the intergenerational effects of individuals. Such a factor holds considerable importance and has been incorporated by various studies such as De la Croiz & Doepke (2003), Gourdel et.al (2004), Currais et.al (2009).

Thus, in order to analyze these effects on human capital accumulation, a two time period OLG model is developed to study the competing nature of health and education. The framework of our study has important implications for the economy. The analysis presents the existence of three steady-state level of equilibrium with one being a poverty-trap. Economies trapped in this state are characterized by high fertily, low human capital and an undernourished population. The medium steady-state is considered to be desireable for the economy in our model since it is characterized by high human capital, low fertility and high child nutrition as opposed to the highest steady-state which is characterized by higher fertility levels, high nutrition and high human capital. However, one of the reasons for higher fertility in the highest steady-state could be stated to be that with an increase in human capital and consequently income, parents have sufficient resources and hence consider the child bearing cost as insufficient and thus are able to rear a large number of children and simultaneously provide them with education and nourishment. Thus, minimum level of fertility in the model is attained when the highest threshold level of human capital $\bar{h}_t$ is reached. Additionally, our framework of analysis has important policy implications for child labor.

The rest of the paper is organized as follows: Section 3 describes the model economy and sets up the framework for analysis. Section 4 states the parents decision problem and provides optimal solutions. Section 5 provides a comparative static analysis and Section 6 concludes.

# 3 Methodological Framework: Model Setup

This section describes the methodological framework for analysis. The model assumes that individuals are homogeneous having identical preferences within each generation. Every individual is equipped with a unitary time endowment in each period.

## 3.1 Households

Time is discrete and goes from 0 to $\infty$. The economy is characterized by an overlapping generations of two periods. In time period 1, individual is young and in time period 2, individual is an adult. In period 1, the young are subcategorized as $(i)$*infants* (time period when the child can neither work nor acquire education) and $(ii)$*young adults*. The adults are the decision makers in each period. In the first time period, a child is equipped with a unitary time endowment which can either be allocated towards work or acquiring education. It is further assumed that the children cannot work or dedicate time to education throughout their lifetime. A fraction $\phi$ of childhood is spent as infants, when the child neither works nor acquires education. Once the infants reach the young adulthood stage, they dedicate their unit of time either in accumulating human capital by acquiring education, $e_{t+1}$ or they are enforced to engage in child labor, denoted as $(1 - e_{t+1} - \phi)$, for the purpose of financing the family consumption, depending on their parents decision.

In the second time period, the young adults of the first time period reach the adulthood stage, beget a child and undertake three crucial decisions. The parents decide:

$(i)$ the optimal level of family consumption $c_t$

$(ii)$ the number of children $n_{t+1}$ to produce and

$(iii)$ the amount of time the child should allocate towards either education or work.

## 3.2 Preferences and the Budget Constraint

In this setup, parents take decisions regarding consumption $c_t$, the number of children $n_{t+1}$ and the proportion of time that the child should devote to education $e_{t+1}$. Parents have identical preferences with respect to the level of household consumption $c_t$ and the level of human capital of the child $H_{t+1}$.These preferences are defined in a log utility function:

$$U_t = \gamma \log c_t + \beta \log n_{t+1} H_{t+1} \tag{3.2.1}$$

where $U_t$ symbolizes the utility function of the household in time period $t$, $\gamma > 0$ represents parental preference for consumption and $\beta > 0$ is the parental altruism factor which shows the extent to which parents care about the childs human capital. The utility function specifies that parents receive maximum utility from consumption $c_t$ and most importantly from both the quantity and quality of their children denoted by $n_{t+1}H_{t+1}$. In this model, education and health both emerge as key mechanisms for a child to accumulate human capital whereby health is evaluated through child nutrition $N_t$. This component is defined as caloric consumption of a child measured in terms of the per child consumption of a household, $\frac{c_t}{n_{t+1}}$. Thus, a child who is devoting a fraction of his/her total time towards education $e_{t+1}$ and receiving nutrition $N_t$ would accumulate human capital $H_{t+1}$ according to the following function:

$$H_{t+1} = N_t^{\alpha_1} \left(e_{t+1} + \theta\right)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2} \tag{3.2.2}$$

where $h_t$ is the inherited human capital. The human capital accumulation technology is in a form to ensure that $H_{t+1}$ is not equivalent to zero. The inclusion of the variable $\theta$ allows human capital to remain positive if the child devotes none of its time towards education (i.e. $e_{t+1} = 0$). In such a scenario, the human capital is low and takes on a value closer to zero. It is included on the assumption that even when time dedicated to education is zero, an individual has some basic skills captured by $\theta$ acquired mainly through informal schooling which can be increased if more time is allocated towards education acquired through formal schooling. Such an assumption has been the basis of many papers (see for example de la Croix & Doepke, 2003; Rivera, Currais & Rungo, 2006; Vasilakis,2011).

Furthermore, child nutrition is a function of per child consumption (pcc$_t$) and maximum nutrition N$_{MAX}$:

$$N_t = Min\left[(pcc_t), N_{MAX}\right] = Min\left[\left(\frac{c_t}{n_{t+1}}\right), N_{MAX}\right] \tag{3.2.3}$$

where $pcc_t$ is the ratio of household consumption $c_t$ to the number of children $n_{t+1}$.

Substituting the function of $N_t$ in $H_{t+1}$ gives the following human capital accumulation technology:

$$H_{t+1} = \left(\frac{c_t}{n_{t+1}}\right)^{\alpha_1} \left(e_{t+1} + \theta\right)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2} \tag{3.2.4}$$

$\alpha_1$ and $\alpha_2$ in the human capital function shows the significance of nutrition and the childs time devoted to education respectively, in formulating its human capital whereas $(1 - \alpha_1 - \alpha_2)$ captures the importance of parents/inherited human capital in determining $H_{t+1}$.

Additionally, the model assumes that time is a constraint faced by adults in deciding the number of children to have. Hence, adults are endowed with one unit of time which is divided between child rearing and working. Bringing up and caring for a child takes a fraction $\phi \in (0, 1)$ of an adults time. Thus, $(1 - \phi n_{t+1})$ is the time allocated by an adult towards working, earning an income according to its human capital $w_t h_t$ where $w_t$ is the wage per efficiency unit of labor and $h_t$ is the parents/adult human capital. Moreover, as is common in the literature (de la Croix & Doepke, 2003; Rivera, Currais & Rungo, 2006) the presence of an opportunity cost of child rearing $w_t h_t \phi n_{t+1}$ leads to a trade-off between the quality and quantity of children. Thus, because rearing each child is time consuming, parents with high wages and human capital find it more costly to have many children. Consequently, such parents prefer quality of children as opposed to quantity and opt to have less but with more education.

Moreover, the children too are endowed with one unit of time which they either allocate towards learning $e_{t+1}$ or working $(1 - e_{t+1} - \phi)$. Again under such a scenario parents face two costs, a direct cost incurred as an expense when providing education (since a private market for education is being assumed) denoted in the model as $e_{t+1} n_{t+1} h_t$. Since education is provided by teachers, all individuals incur a cost per child equivalent to the average human capital of the teacher which equals the average human capital of the population. In a homogeneous population the average human capital of the teacher is equivalent to the adult human capital $h_t$. Thus, the total cost of education incurred is $e_{t+1} n_{t+1} h_{t,}$. The second cost incurred is an opportunity cost when the child devotes time to education instead of working denoted as $k e_{t+1} n_{t+1}$. In the case where a child does not work, $(1 - \phi)$ represents the maximum time spent on education. When a child works it earns an amount $k$. It is assumed that this child labor wage cannot be higher than the cost of child rearing i.e.

$$k < \phi w_t h_t \tag{3.2.5}$$

Therefore, the consumption profile $c_t$ is as follows:

$$c_t = h_t\left(1 - \phi n_{t+1}\right) + k\left(1 - e_{t+1} - \phi\right) n_{t+1} - e_{t+1} n_{t+1} h_t \tag{3.2.6}$$

where $w_t$ has been normalized to one for simplicity.

## 3.3 Production

It is assumed that only one good is produced in the economy. Production is linear in nature, entails using a constant returns to scale technology and is solely dependent on the human capital of the individual.

$$Y_t = H_t \tag{3.3.1}$$

where $H_t$ is the total human capital of the workforce and is a function of the following at the labor market equilibrium:

$$H_t = h_t\left(1 - \phi n_{t+1}\right) + k\left(1 - e_{t+1} - \phi\right) n_{t+1} \tag{3.3.2}$$

Since a competitive market exists, hence at the equilibrium point wages are equal to the childrens marginal productivity and are normalized to one for simplification purposes.

# 4 Parents Decision Problem

Parents jointly determine the number of children $n_{t+1}$ to have and the proportion of time that the child should spend on education, $e_{t+1}$. Thus, the household maximization problem is:

$$\max_{(e_{t+1},n_{t+1})} U_t = \gamma \log c_t + \beta \log n_{t+1} H_{t+1} \tag{4.1}$$

subject to the constraints:

$$H_{t+1} = \left(\frac{c_t}{n_{t+1}}\right)^{\alpha_1} (e_{t+1} + \theta)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$

$$c_t = h_t (1 - \phi n_{t+1}) + k (1 - e_{t+1} - \phi) n_{t+1} - e_{t+1} n_{t+1} h_t$$

$$Y_t = H_t$$

Incorporating the constraints in (4.1) yield the following household value function:

$$V(e_{t+1}, n_{t+1}) = (\gamma + \beta\alpha_1) \log [h_t (1 - \phi n_{t+1}) + k n_{t+1} (1 - \phi) - e_{t+1} n_{t+1} (h_t + k)]$$
$$+ \beta\alpha_2 \log (e_{t+1} + \theta) + \beta (1 - \alpha_1) \log n_{t+1} + \beta (1 - \alpha_1 - \alpha_2) \log h_t \tag{4.2}$$

The parents maximize this value function with respect to $e_{t+1}$ and $n_{t+1}$ to yield the following first order conditions:

$$\frac{\partial V}{\partial e_{t+1}} = \frac{(\gamma + \beta\alpha_1)}{[h_t (1 - \phi n_{t+1}) + k n_{t+1} (1 - \phi) - e_{t+1} n_{t+1} (h_t + k)]} [-n_{t+1} (h_t + k)] + \frac{\beta\alpha_2}{(e_{t+1} + \theta)} = 0 \tag{4.3}$$

$$\frac{\partial V}{\partial n_{t+1}} = \frac{(\gamma + \beta\alpha_1)}{[h_t (1 - \phi n_{t+1}) + k n_{t+1} (1 - \phi) - e_{t+1} n_{t+1} (h_t + k)]} [-\phi h_t + k (1 - \phi) - e_{t+1} (h_t + k)]$$
$$+ \frac{\beta (1 - \alpha_1)}{n_{t+1}} = 0 \tag{4.4}$$

The first order conditions gives the following two equations which shows the interrelationship between the two choice variables with respect to each other as well as human capital $h_t$:

$$n_{t+1} = \frac{\beta\alpha_2 h_t}{e_{t+1}\left(h_t+k\right)\left(\gamma+\beta\alpha_1+\beta\alpha_2\right)+\theta\left(\gamma+\beta\alpha_1\right)\left(h_t+k\right)+\beta\alpha_2\left[\phi h_t-k\left(1-\phi\right)\right]} \tag{4.5}$$

$$e_{t+1} = \left[\frac{\beta\alpha_2\left[h_t\left(1-\phi n_{t+1}\right)+kn_{t+1}\left(1-\phi\right)\right]}{n_{t+1}\left(h_t+k\right)\left(\gamma+\beta\alpha_1\right)}-\theta\right]\left[\frac{\left(\gamma+\beta\alpha_1\right)}{\left(1+\gamma+\beta\alpha_1\right)}\right] \tag{4.6}$$

Eq.(13) postulates an inverse relation with both $e_{t+1}$ and $h_t$, entailing a declining fertility which is motivated due to a shift in parents preferences from child quantity towards quality augmented through parental investment in education. Eq.(14) shows a positive relationship of $e_{t+1}$ with $h_t$ implying that the greater the human capital of the parents in the form of higher knowledge and skills, the more their child would spend time on education enhancing its intellectual development. The result reinforces the concept that a child is more likely to be sent to school if parents are educated and that human capital is an important factor in intergenerational mobility. When analyzing $e_{t+1}$ with respect to $n_{t+1}$, a negative relationship is observed, complying with contemporary research that parents face a trade-off between the number of children to have and the time devoted to education. A decision to have more children implies that education becomes relatively expensive and constrains the parents resources, hence the decision to substitute quality with quantity. Therefore, $h_t$ plays an important role in shifting (i) parents preference away from fertility and towards working due to a rise in the opportunity cost of having a child and (ii) the child's time towards education rather than working and in the process deterring child labor.

Using these first order conditions provides the optimal solutions for $e_{t+1}$ and $n_{t+1}$(refer to Appendix A):

$$e^* = \begin{cases} 0 & \frac{k}{\phi} \leq h_t \leq \underline{h}_t \\ \frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)} & \underline{h}_t < h_t < \bar{h}_t \\ 1-\phi & h_t \geq \bar{h}_t \end{cases} \tag{4.7}$$

$$n^* = \begin{cases} \frac{\beta\alpha_2 h_t}{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k} & \frac{k}{\phi} \leq h_t \leq \underline{h}_t \\ \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]} & \underline{h}_t < h_t < \bar{h}_t \\ \frac{\beta\alpha_2 h_t}{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k} & h_t \geq \bar{h}_t \end{cases} \tag{4.8}$$

Incorporating the optimal values $e^*$ and $n^*$ in $N_t = \frac{c_t}{n_{t+1}}$ gives the optimal solution for child nutrition:

$$N^* = \begin{cases} \frac{\theta(\gamma+\beta\alpha_1)(h_t+k)}{\beta\alpha_2} & \frac{k}{\phi} \leq h_t \leq \underline{h}_t \\ \frac{(\gamma+\beta\alpha_1)[(h_t+k)(\phi-\theta)-k]}{\beta(1-\alpha_1-\alpha_2)} & \underline{h}_t < h_t < \bar{h}_t \\ \frac{(\gamma+\beta\alpha_1)(h_t+k)(1+\theta-\phi)}{\beta\alpha_2} & h_t \geq \bar{h}_t \end{cases} \tag{4.9}$$

where

$$\underline{h}_t = \frac{k\left[\alpha_2\left(1-\phi\right)+\theta\left(1-\alpha_1\right)\right]}{\alpha_2\phi-\theta\left(1-\alpha_1\right)} \tag{4.10}$$

$$\bar{h}_t = \frac{k\left(1-\alpha_1\right)\left(1+\theta-\phi\right)}{\left[\alpha_2-\left(1-\alpha_1\right)\left(1+\theta-\phi\right)\right]} \tag{4.11}$$

$\underline{h}_t$ and $\bar{h}_t$ are the low and high human capital respectively and define two distinct regimes in the economy. The first regime with low human capital $\underline{h}_t$ is characterized by an economy with pure child labor i.e. children do not dedicate any time or effort towards education. After the infancy stage, their whole life is dedicated to earning in order to help the family in increasing the income for survival. $\bar{h}_t$ in contrast, defines an economy where there is no child labor. All the children dedicate their maximum time towards education. An interior economy with ($\underline{h}_t < h_t < \bar{h}_t$) comprises a scenario where both child labor and education simultaneously exist.

The way the economy behaves with respect to the optimal variables over time in the three regimes is depicted in Figure 1(refer to Appendix C):

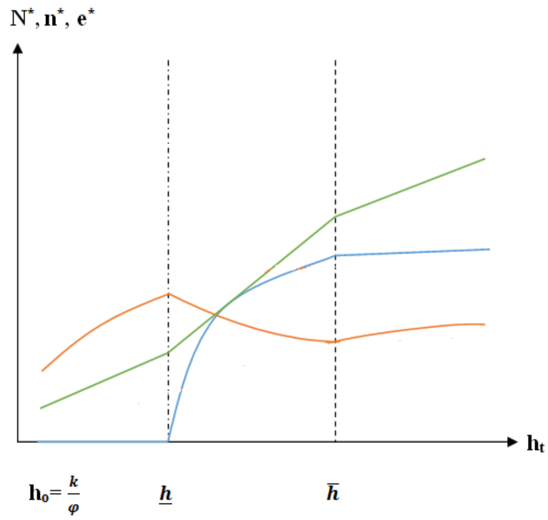



Figure 1: Education, Fertility & Child Nutrition

In order to ensure that the slopes remain positive and lie within the specified range, $\phi$ is needed to be bounded (refer to Appendix C). :

$$\phi \leq \frac{\beta\alpha_2 (1-\alpha_1) + (\gamma + \beta\alpha_1)(1-\alpha_1-\alpha_2)}{(1-\alpha_1)(\gamma+\beta\alpha_1+\beta\alpha_2)} \equiv \tilde{\phi} \tag{4.12}$$

Inequality (4.12) ensures that $\phi$ and $\theta$ are both less than $\tilde{\phi}$.

It also ensures that $\theta$ lies in the follwing interval:

$$\theta_B \approx \phi - \frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)} < \theta < \frac{\beta\alpha_2 (1-\phi)}{(\gamma+\beta\alpha_1)} \approx \theta_A \tag{4.13}$$

Inequality (4.13) shows the upper and lower limits for the minimum level of skills required by a child even when it receives no education.

Figure 1 shows the behaviour of education, fertility and child nutrition in the model. They are denoted through different colors. Education is denoted by blue line, fertility through the red line and child nutrition trhough the green line.

The figure shows that for low levels of human capital i.e. when $\frac{k}{\phi} \leq h_t \leq \underline{h}_t$, time dedicated to education is zero. This implies that none of the children go to school instead they devote the whole of their available time towards labor. Child nutrition and fertility under this regime both gradually increase. Parents are poor in this regime having less resources and thus bear a large number of children all of whom are both uneducated as well as receive less nourishment. The intuition behind parents preferring to bear and rear more children is to help them in meeting their family and consumption needs by working in the child labor market and earining a certain income. Thus, the reason for parents to shift their childs time away from education and towards working and the main reason for education being equivalent to zero in this regime. Since fertility is high and resources are low, hence less resources are available for the nourishment of a child. Thus the economy is characterized by a large number of undernourished children.

Moreover, under such a scenario parents prefer quantity as compared to quality of children. The marginal rate of substitution is also lower as compared to the relative price of child quantity to the price of child quality. Thus parents find rearing and bearing children less costly and respond by preferring to have more children in order to increase resources and family income. When $h_t \rightarrow \underline{h}_t$ the maximum level of fertility is attained after which it declines with an increase in income.

When $h_t > \underline{h}_t$, both child labor and education simultaneously exist. The time dedicated to education is an increasing concave down in this interval i.e. $\partial e_{t+1}/\partial h_t > 0$ . This indicates that time devoted towards education increases but at a decreasing rate. Resources devoted towards nourishing too increases but at a constant rate while fertility declines at an increasing rate. During this interval, parents engage in a trade-off and hence prefer child quality as opposed to child quantity which marks the beginning of the demographic transition whereby the economy moves from high birth rates to low birth rates.

At higher levels of human capital, above the threshold level $\bar{h}_t$, the time dedicated to education reaches an upperbound level which is its maximum level $(1 - \phi)$ . The non-increasing nature of education at this level is an indication that resources are being devoted towards child nutrition. Hence, as can be observed from Figure 1, child nutrition is increasing at a constant rate. However it is less steep than before. This is due to the fact that maximum consumption of nutrition has been reached at $\bar{h}_t$ and therefore execissive consumption above the threshold level has little impact on nutrition or health status. Additionally, in the model when nutrition and education reach their maximum level, the number of children cease to decrease. Rather an increase in fertility is observed. This result is counterintuitive and puzzling and I have no plausible intution as to why fertility is increasing with an increase in human capital. The derivatives also show an increasing slope. However, in accordance to my knowledge, such an observation could be explained by some of the studies who are cosistent with my result of increasing fertility at the highest level of human capital attained. According to contemporary literature, there is a fertility rebound or fertility reversal occuring in most developed nations. Klüsene, Fox, & Myrskyla analyzed the data of 19 European countries from 1996 to 2010. Their study found mixed results with respect to the income-fertility correlation. For Spain the correlation moved from negative to positive. Poland, Croatia and Portugal experienced positive correlations throughout. They further investigated the relationship by using a panel model to control for country-level differences and country-specific period effects in fertility. The result showed a convex relationship between income and fertility for both West and East Europe. Luci & Thévenon (2010) also find similar results but for OECD countries. Varvarigos (2013) also supports such an argument by using an OLG model, the authors analysis reveals an N-shaped fertility curve. Luci-Greulich & Thévenon (2014), Vogl (2013), Dominiak, Lechman, & Okonowicz (2014),Goldstein & Sobotka (2009), Frejka (2010) are some studies which corroborate with the fertility reversal trends in developed nations.

Additionally, looking at the results, one of the reasons for higher fertility could be stated to be that with an increase in human capital and consequently income, parents have sufficient resources and hence consider the child bearing cost as insufficient and thus are able to rear a large number of children and simultaneously provide them with education and nourishment.

Thus, so far the analysis reveals that in the intermediate phase, the economy experiences a demographic transition. In this interval the child quantity-quality trade-off is observed but at a later stage the continuing process of increasing human capital allows agents to generate adequate resources to rear more children and simultaneously endowing them with the capacity of providing the children with education and nutrition. Minimum level of fertility in the model is attained when maximum child nutrition and time devoted towards education are attained i.e. when human capital reaches its highest threshold level. When human capital exceeds the maximum threshold level i.e. when $h_t > \bar{h}_t$, time devoted to education becomes a constant i.e. further increases in human capital has no effect on the time devoted to education pertaining that maximum capacity to learn has been reached.

## 4.1 Evolution of Human Capital and The Steady-State

Human Capital evolves according to the following function when the optimal solution of $e^*, n^*$ and $N^*$ of all three regimes are plugged into it:

$$H_{t+1} = \begin{cases} \frac{\theta^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}(h_t+k)^{\alpha_1}h_t^{1-\alpha_1-\alpha_2}}{(\beta\alpha_2)^{\alpha_1}} & \frac{k}{\phi} \leq h_t \leq \underline{h}_t \\ \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}[\phi h_t - k(1-\phi)-\theta(h_t+k)]^{\alpha_1+\alpha_2}h_t^{1-\alpha_1-\alpha_2}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}(h_t+k)^{\alpha_2}} & \underline{h}_t < h_t < \bar{h}_t \\ \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}(h_t+k)^{\alpha_1}h_t^{1-\alpha_1-\alpha_2}}{(\beta\alpha_2)^{\alpha_1}} & h_t \geq \bar{h}_t \end{cases} \quad (4.1.1)$$

The human capital accumulation results in the following three steady-states:

$$h^*_{s1} = \left[\frac{(\gamma+\beta\alpha_1)\,k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}} \theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}} \quad (4.1.2)$$

$$h^*_{s2} = \left[\frac{[\phi - k\,(1-\phi) - \theta\,(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}} \left[\frac{\alpha_2^{\alpha_2}\,(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}} \quad (4.1.3)$$

$$h^*_{s3} = \left[\frac{(\gamma+\beta\alpha_1)\,k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}} (1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}} \quad (4.1.4)$$

The steady-states of the respective regimes are denoted as follows:

(i) $h_{s1}$ as $h_L$.

(ii) $h_{s2}$ as $h_m$.

(iii) $h_{s3}$ as $h_H$.

The following figure shows the existence of three steady-states:

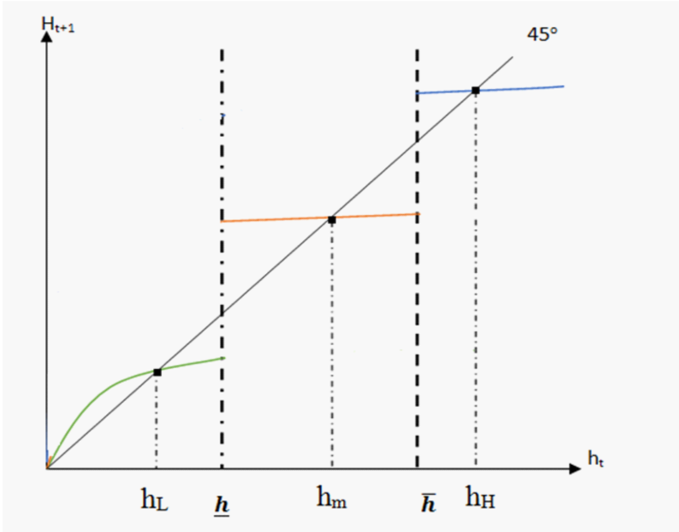



Figure 2: Human Capital & The Steady-State

*Proposition 1*

*1) When individuals initial stock of human capital $h_t$ lies in the interval $(0, h_L)$, the economy converges to the steady-state equilibrium $h_L$ by increasing its stock of human capital.*

*2) When individuals initial stock of human capital $h_t$ lies in the interval $\left(h_L, \underline{h}_t\right)$, the economy converges to either a lower steady-state $h_L$ or a relatively higher steady-state $h_m$ depending on parental preferences. Hence, if parental preferences for child quantity is high relative to child quality, the economy's stock of human capital would decline to the steady-state $h_L$, thus falling into a poverty-trap. If parental preferences for child quality outweighs child quantity, the economy would converge to a relatively higher steady-state $h_m$.*

*3) When individuals stock of human capital $h_t$ lies in the interval $\left(h_m, \bar{h}_t\right)$, the economy either converges to a medium level of steady-state $h_m$ or converges to the highest steady-state equilibrium $h_H$, depending on parental preferences for child quantity or quality. Thus if child quality outweighs child quantity, the economy would converge to the highest steady-state equilibrium $h_H$.*

Proposition 1 states the possibility of three steady-state level of equilibrium for an economy to converge. Convergence to the three different steady-states depends firstly on the initial stock of human capital and secondly on parental preferences for either child quantity or child quality. Thus, if initial stock of human capital lies in the interval $(0, h_L)$, the economy converges to $h_L$ characterized by high fertility, low nutrition, low human capital and high level of child labor. However, if the initial stock of human capital lies in the interval $(h_L, \underline{h}_t)$, the economy has the option to either converge to a low steady-state $h_L$ or to converge towards a higher steady-state $h_m$. Convergence under such a scenario depends on parental preferences for child quantity or child quality. Thus if child quality outweighs child quantity, the time of a child shifts towards education rather than working and in the process discourages child labor. Parents preferences also shift away from fertility and towards working due to a rise in the opportunity cost of having children. Hence, the economy converges to a higher steady-state $h_m$, characterized by low fertility, high nutrition and human capital. However, if child quantity outweighs child quality, the time of a child shifts away from education and towards working. Thus, human capital would decrease and the economy would converge to the steady-state $h_L$ characterized by a high level of child labor, higher level of fertility and low nutrition levels. Thus, the economy would be trapped in poverty and consist of high levels of uneducated and undernourished population. Conversely, if stock of human capital is in the interval $\left(h_m, \bar{h}_t\right)$, the economy could either converge to the highest steady-state $h_H$ or to a medium steady-state $h_m$ dependent on parental preferences.

In order to ensure that the steady-states remain within the range specified i.e.

$$h_L < \underline{h}_t$$

$$\underline{h}_t < h_m < \bar{h}_t$$

$$h_H > \bar{h}_t$$

it is a necessary condition that:

$$k < \left[ \frac{\beta^{\frac{\left(5\alpha_1^3 - 40\alpha_1^2\alpha_2 - 45\alpha_1\alpha_2^2\right)}{(2\alpha_1 - 2\alpha_2)}} \alpha_2^{\frac{1}{2}(4\alpha_1 - 19\alpha_2)(\alpha_1+\alpha_2)}}{(\gamma+\beta\alpha_1)^{\frac{5}{2}\alpha_1\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}(\alpha_1-9\alpha_2)}(1-\alpha_1)^{\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}\left(34\alpha_2^2-\alpha_1^2+7\alpha_1\alpha_2\right)} (1-\alpha_1-\alpha_2)^{\frac{1}{2}(\alpha_1+\alpha_2)(\alpha_1-11\alpha_2)}} \right]^{\frac{1}{\alpha_1^2-\alpha_2^2}} \equiv \tilde{k} \qquad (4.1.5)$$

where $k$ is the child labor wage. The condition (4.1.5) has important policy implications for child labor (refer to Appendix E).

*Proposition 2*

*1) When child labor wage is less than the threshold level i.e.* $k < \tilde{k}$*, parents do not have an incentive to send their children to work, child labor declines, schooling increases thereby increasing the human capital of the child. The economy thus converges to a high steady-state* $h_H$*, characterized by a higher level of fertility and an educated and healthy population.*

*2) When child labor wage is greater than the threshold level i.e.* $k > \tilde{k}$*, parents have an incentive to send their children to work, child labor increases, schooling levels decline resulting in a lower level of human capital of the child. The economy thus converges to a low steady-state* $h_L$*, characterized by a higher level of fertility and an uneducated and undernourished economy.*

*3) When child labor wage is equivalent to the threshold level i.e.* $k = \tilde{k}$*, parents are indifferent between sending their child either to school or for work or engage in both simultaneously. Hence, both high and low human capital level of children would persist in the economy. Thus, the economy would converge to an unstable equilibrium* $h_m$ *which is characterized by low fertility and high nutrition levels.*

The economic intuition behind Proposition 2 is that since $\tilde{k}$ represents the minimum level of wage, hence when children receive a lower wage in the market compared to the threshold level, parents have less incentive to send their children to work since one of the main reason for sending their children off to work is for them to contribute more towards the family income in order to meet their consumption needs. However, a lower child labor wage would be insufficient to meet their consumption needs, discouraging parents to send their child to work. This then acts as an incentive for the children to devote more of their time towards education. As a result child labor in the economy would decline and consequently, human capital of the child would increase, allowing an economy to develop and move towards a higher level of steady-state which in the model developed is characterized by a higher level of fertility as well as a higher level of education and health status. A vice versa situation would persist in the case of child labor wage being greater than the threshold level. When $k > \tilde{k}$, since children are receiving a higher level of wage in the market when offering their services, thus it would act as an incentive for the parents to send their children to work instead of school in order for them to contribute towards the consumption needs of the family. As a result, such an action would shift the childs time away from schooling and more towards work. Child labor in such a scenario would increase and human capital would consequently decline. Since lower level of human capital would persist, thus the economy would converge to a low steady-state which is infact a poverty-trap and is characterized by an economy with high fertiliy, uneducated and undernourished population.

When $k = \tilde{k}$, parents are indifferent between sending their child to school or work. Thus, in such an economy, parents depending on their resources would make such a decision. Hence, both low and high human capital levels would persist in the economy and the economy would converge to a medium steady-state, $h_m$, which is characterized by low levels of fertility and high nutrition levels.

Thus, overall the results suggest that in order to discourage child labor, the child labour wage should be below the threshold level which would allow the economy to converge to the high steady-state since parents would prefer that their child spend its time on learning rather than working in the labor market. This would then induce an increase in the human capital of the child.

# 5 Comparative Statistics

In this section we analyze the effects of child labor wage $k$ and the impact of minimum skills level inherent in a child when it receives no formal schooling denoted by $\theta$ on the economy by conducting a comparative static analysis in the steady-state. We analyze their effects by looking at the derivatives, the effects of changing child labor wage and minimum skills on each of the three steady-state levels of time devoted to education, child nutrition and fertility (refer to Appendix F).

The steady-state values of $e^*, n^*$ and $N^*$ are as follows where $sL$ denotes the optimal value in low steady-state $h_L$, $sm$ denotes the optimal value in medium steady-state $h_m$ and $sH$ denotes the optimal value in high steady-state.

The steady-state values of $e^*$ are as follows:

$$e^*_{sL} = 0 \tag{5.1}$$

$$e^*_{sm} = \left( \frac{\alpha_2^{2\alpha_1+3\alpha_2}\theta^{\alpha_1}\left(1-\alpha_1-\alpha_2\right)^{\alpha_1+\alpha_2}\beta^{\alpha_1}}{\phi^{\alpha_1}\left(\gamma+\beta\alpha_1\right)^{\alpha_1}} \right)^{\frac{1}{\alpha_1+2\alpha_2}} \left( \frac{k}{(1-\alpha_1)} \right) \tag{5.2}$$

$$e^*_{sH} = 1-\phi \tag{5.3}$$

The steady-state values of $N^*$ are as follows:

$$N^*_{sL} = \left[ \frac{\theta^{(\alpha_1+3\alpha_2)}k^{(\alpha_1+\alpha_2)}\left(\gamma+\beta\alpha_1\right)^{(\alpha_1+2\alpha_2)}}{\left(\beta\alpha_2\right)^{(\alpha_1+2\alpha_2)}} \right]^{\frac{1}{\alpha_2}} \tag{5.4}$$

$$N^*_{sm} = \left[ \frac{\left(\gamma+\beta\alpha_1\right)^{\alpha_1}\phi^{(4\alpha_1+6\alpha_2)}\alpha_2^{\alpha_2}}{\theta^{3\alpha_1+5\alpha_2}\beta^{\alpha_1}k^{2\alpha_1+3\alpha_2}\left(1-\alpha_1-\alpha_2\right)^{\alpha_1+\alpha_2}} \right]^{\frac{1}{\alpha_1+2\alpha_2}} \tag{5.5}$$

$$N^*_{sH} = \left[ \frac{\theta^{\alpha_1+2\alpha_2}\left(\gamma+\beta\alpha_1\right)^{\alpha_1+\alpha_2}k^{\alpha_1+\alpha_2}}{\left(\beta\alpha_2\right)^{\alpha_1+\alpha_2}\phi^{\alpha_1+2\alpha_2}} \right]^{\frac{1}{\alpha_2}} \tag{5.6}$$

The steady-state values of $n^*$ are as follows:

$$n^*_{sL} = \frac{(\beta\alpha_2)}{\phi\theta\,(\gamma + \beta\alpha_1)} \tag{5.7}$$

$$n^*_{sm} = \frac{(\gamma + \beta\alpha_1)\,\theta\beta\,(1 - \alpha_1 - \alpha_2)\,\phi^{\frac{\alpha_1}{\alpha_1+2\alpha_2}}}{k^{\left(2\frac{(\alpha_1+\alpha_2)}{\alpha_1+2\alpha_2}\right)}\,(\gamma + \beta)} \tag{5.8}$$

$$n^*_{s_H} = \left[\frac{1}{(1 - \phi + \theta)^{(\alpha_1+2\alpha_2)}\,(\gamma + \beta\alpha_1)^{(\alpha_1+\alpha_2)}\,(\beta\alpha_2)^{(\alpha_1-2\alpha_2)}\,k^{(\alpha_1+\alpha_2)}}\right]^{\frac{1}{\alpha_2}} \tag{5.9}$$

We conduct a comparative static analysis of equations (5.1) to (5.9). .

For equation (5.1) and (5.3) the derivatives with respect to $k$ and $\theta$ are zero since the steady-state values are not a function of these variables. Rather it is a constant value.

The following shows a comprehensive review of the comparative statistics conducted (refer to Appendix G for proof).

$$\frac{\partial e^*_{s_L}}{\partial k} = 0 \qquad \frac{\partial e^*_{sm}}{\partial k} > 0 \qquad \frac{\partial e^*_{s_H}}{\partial k} = 0 \tag{5.10a}$$

$$\frac{\partial e^*_{s_L}}{\partial \theta} = 0 \qquad \frac{\partial e^*_{sm}}{\partial \theta} > 0 \qquad \frac{\partial e^*_{s_H}}{\partial \theta} = 0 \tag{5.10b}$$

$$\frac{\partial N^*_{s_L}}{\partial k} > 0 \qquad \frac{N^*_{sm}}{\partial k} < 0 \qquad \frac{\partial N^*_{s_H}}{\partial k} > 0 \tag{5.11a}$$

$$\frac{\partial N^*_{s_L}}{\partial \theta} > 0 \qquad \frac{\partial N^*_{sm}}{\partial \theta} < 0 \qquad \frac{\partial N^*_{s_H}}{\partial \theta} > 0 \tag{5.11b}$$

$$\frac{\partial n^*_{s_L}}{\partial k} = 0 \qquad \frac{\partial n^*_{sm}}{\partial k} < 0 \qquad \frac{\partial n^*_{s_H}}{\partial k} < 0 \tag{5.12a}$$

$$\frac{\partial n^*_{s_L}}{\partial \theta} < 0 \qquad \frac{\partial n^*_{sm}}{\partial \theta} > 0 \qquad \frac{\partial n^*_{s_H}}{\partial \theta} < 0 \tag{5.12b}$$

*Proposition 3*

*i) An increase in the child labor wage (i.e. a higher $k$) and minimum skills level (i.e. a higher $\theta$ ) leads to a higher level of low steady-state level of child nutrition.*

*ii) A higher $\theta$ reduces the low steady-state level of fertility but a higher $k$ induces no effect on it.*

*iii) An increase in the child's wage (a higher $k$) increases the medium steady-state level of time spent on education but lowers the medium steady-state level of nutrition and fertility.*

*iv) An increase in the minimum skills level (a higher $\theta$) leads to a higher medium steady-state level of time spent on education and fertility but lowers the medium steady-state level of child nutrition.*

*v) An increase in both the child's wage and minimum skills level leads to an increase in the higher steady-state level of child nutrition and a decrease in the higher steady-state level of fertility*

Proposition 3 states the relationship of $k$ and $\theta$ with the steady-state values. It can be observed from (5.10a and 5.10b) that regardless of a change in the child labor wage and the minimum level of skills, the low steady-state level of time devoted to education would be zero. This is because the economy is characterized by pure child labor. Hence, regardless of an increase or decrease in child labor wage and regardless of the skill level inherent in a child, parents would send their children in the labor market to sell their services to fulfill the consumption needs of the family. Similarly, regardless of a change in $k$ and $\theta$ the high steady-state level of $e^*$ would be a maximum. Children would spend all their time after infancy in devoting to education rather than working. However, for the medium steady-state level, time spent on education increases with an increase in $k$. Such a result seems plausible since the economy in this state is mixed i.e. children work and study simultaneously. The increase induced by $k$ on $e^*_{sm}$ could be due to the fact that children earning a wage might to some extent be able to finance their education and thus the increase in the time spent on learning. When analyzing $\theta$, it exerts a positive impact on $e^*_{sm}$. The intuition behind such a result could be that when minimum skill level of a child increases, it's capacity to learn increases and hence it utlizes such a trait by gaining more knowledge, thus the rise in time devoted to education.

Observing (5.11a and 5.11b), an increase in $k$ and $\theta$ induces a positive impact on child nutrition in the low steady-state. A credible argument for such an impact could be that since the low steady-state is characterized by pure child labor, thus, because all of the time is being spent by a child on working, it receives wages for offering its services which could be used by the child for its own nourishment. Such an impact is further induced by an increase in the minimum skills level $\theta$ which might be gained for example through learning on the job in the unskilled sector. A rise in $\theta$ allows the child to enhance its capability and productivity in the unskilled labor market, and thereby allowing it to receive a higher wage than the minimum, which in turn could be further used by the child for its own nourishment.

For the medium steady-state level, child nutrition decreases with an increase in $k$ and $\theta$. The intuition behind such a result could be analyzed by observing the medium steady-state level of time devoted to education given in (5.10a and 5.10b) which increases with $k$ and $\theta$. The decrease in child nutrition in the medium steady-state level could be attributed to the fact that the income earned through child labor might be being used for financing the child's education and thus less resources are left for food consumption. Moreover, the option for letting the child obtain education could be due to a shift in parental preferences away from child quantity and towards child quality (observe the medium steady-state level of fertility in 5.12a which reinforces this statement). Similar economic interpretation could be given for $\theta$. When the minimum skills level increases in the medium steady-state level, child nutrition declines since resources and time both are being spent on education (observe medium steady-state level of time devoted to education in 5.10a which shows an increase with $\theta$). However, fertility increases with a rise in $\theta$. The result seems unambiguous however this rise could also be a reason for the decline in child nutrition. With rising fertility, less of resources are available for the nourishment of each child thus the decline in child nutrition with $\theta$.

When analyzing the high steady-state level of child nutrition and fertility, then the results correspond with each other. With an increase in both $k$ and $\theta$, child nutrition increases whereas fertility declines. The intuition behind such a result could be that in the high steady-state a rise in child labor wage allows the child to spend it on food consumption thereby inducing an increase in child nutrition. Similarly since in the high steady-state, a population with a high skill level is dominant, thus increases in $\theta$, could further enhance the productivity and capability of the child in the labor market allowing it to earn a higher wage and thus use it for receiving nourishment. Moreover, since fertility is declining thus more of resources per child is available for food consumption.

A brief summary of the results is that child labor wage and minimum level of skills inherent in a child has an impact on fertility, child nutrition and on the childs time spent on education. An increase or decrease in both, thus has important economic implications.

# 6 Conclusion

A vast literature exists which emphasizes the importance of health for the productivity and efficiency of human capital and growth however, most of these studies focus on different health measures, diseases or on the efficiency of different types of health sectors. Child nutrition, a key aspect in improving health is overlooked in most theoretical papers. Thus, our study provides a theoretical framework for incorporating both child nutrition and child education as key factors in augmenting the human capital accumulation process. We study the competing effects of both for resources by incorporating the cost of acquiring education in the private market.

The results of our study show that human capital plays an important role in two aspects. First, it helps in shifting the preferences of parents away from child bearing and rearing by increasing the loss of remuneration and thereby influencing their fertility decisions and second in shifting the childs time away from work and towards education.

The analysis yields three steady-state level of equilibrium. Parents having low human capital prefer child quantity over child quality and hence are trapped in the low steady-state where both child labor as well as an undernourished population is dominant. The medium steady-state is considered to be desireable for the economy in our model since it is characterized by high human capital, low fertility and high child nutrition as opposed to the highest steady-state which is characterized by higher fertility levels, high nutrition and high human capital. However, one of the reasons for higher fertility in the highest steady-state could be stated to be that with an increase in human capital and consequently income, parents have sufficient resources and hence consider the child bearing cost as insufficient and thus are able to rear a large number of children and simultaneously provide them with education and nourishment. Thus, minimum level of fertility in the model is attained when the highest threshold level of human capital $\bar{h}_t$ is reached.

The results of our study also has important implications for child labor policies. The analysis indicates that child labor wage is an important factor in either discouraging or encouraging child labor. Thus, a child labor wage below the threshold level in the model would discincentivise parents to send their children to work since one of the main reason for sending their children off to work is for them to contribute more towards the family income in order to meet their consumption needs. However, a lower child labor wage would be insufficient to meet their consumption needs, thus discouraging parents to send their child to work. This then acts as an incentive for the children to devote more of their time towards education. As a result child labor in the economy would decline and consequently, human capital of the child would increase, allowing an economy to develop and move towards a higher level of steady-state. Thus, effective policies could be designed aiming for a sufficiently lower child labor wage to deter parents from sending their child to work.

Also, the study points to the fact that children work, not because there isn't any incentive involved in sending them to school but because of certain limitations that families face in doing so. Such constraints emerge either due to poverty or from temporary shocks. In both cases families are forced to make necessary short-run decisions which turn out to be sub-optimal for the welfare of the children in the long-run. Instruments that can effectively support families in managing risks related to poverty and other forms of vulnerability can reduce or eliminate these obstacles and give families choices beyond resorting to sending their children to work.

Public safety nets are an effective way which can prove as an alternative for families that send children to work. Most countries have some safety net programs in place, although the extent, sophistication, and effectiveness of these programs vary considerably.

Various successful programs around the world are being operated. Public works or welfare programs have been the most widely used type of safety-net intervention in low-income countries. If conducted and designed well, such programs provide income (or food, in some cases) to poor households which help them to smooth their consumption, without resorting to strategies such as child labor.

Conditional transfers represent a type of safety net program which helps in reducing child labor and increasing school attendance. These programs provide either cash or food to poor families with the condition that they send their children to school instead of work. They have been used most widely in Latin America. One of that region's best known conditional cash transfer programs is Mexico's Progresa, which links cash grants and nutritional supplements to school and clinic attendance. The results of the program showed a decrease in child labor and an increase in child schooling.

Bangladesh initiated a Food for Education program, in which poor families were given free monthly ration of rice or wheat on the condition that their children attend primary school. Turkey also initiated a conditional cash transfer program in order to improve both the children's school attendance and their health.

All the aforementioned programs can be categorized as child benefits which have implications on the child's nutrition and its human capital as well as affect the fertility decisions of parents.

First, feeding programs use food as a tool to attract children to school. It is expected that such a program would increase the total level of nutrition of children. Child nutrition would still be an increasing function of human capital. However, such food supplements in school could decrease the amount of nutrition offered from parents to their children in all regimes.

Fertility, however in such a scenario would increase. A trade-off of the child quantity over the child quality would take place. School food supplement programs increase fertility by reducing the "quantity cost" of children, thereby shifting resources from quality to quantity of children. In other words, parents decrease the level of nutrition of their offspring and they increase the number of their children. Such a result is in compliance with Azarnet (2008) where humanitarian aid increases fertility by decreasing parental investment in child education. This consequently results in reducing the children's human capital accumulation. Neanidis (2010) also shows similar results where per adult aid increases fertility by reducing the "quantity cost" of children.

Thus, such programs prove to be useful measures in curbing child labor and increasing school attendance, subsequently resulting in an increase in the human capital accumulation (see Chandler et al, 1995; Chang et al, 1996).

A second form of child benefit is subsidizing the price of nutrition of children. This form of benefit is expected to lead to an improvement in both the nutrition level as well as an improvement in human capital. Neanidis (2010) shows that such a program raises the probability of a child's survival leading to decreased fertility and increased nutrition consequently resulting in positive growth. This type of program subsequently allows the poor developing countries to escape from a poverty trap given that sufficient amount of aid is being provided to them. However, if aid is insufficient then child labor would continue to exist. Fertility would increase and nutritional supplements might not be sufficient to lead to an improvement in the child's health.

Another child benefit could be in the form of food provision to households. In this scenario government or an organization offers a fixed amount of nutrition for each child of a household. An example of such an aid is by WFP in 2010, which provided 36.500 metric tons of food aid to assist families in Pakistan. Such an aid could increase the time devoted towards education since more nutritional supplements per child means more healthy children who are capable of concentrating their time towards education. However, parents own investment in the health of their children decreases. Fertility could increase or remain the same in such a scenario. The freed resources of the parents could make it affordable for parents to bear the cost of an additional child hence they would prefer to have more children in such a scenario however, if the freed resources are used for investment in education of existing children then fertility could remain the same.

A vast literature exists which observes large variations in educational and child labor policies across countries during the transition to growth. Most countries introduce education and child labor reforms at some point during their development, but the extent and timing of these reforms varies widely. A good example is that of Brazil and Korea. Starting in the mid-1950s (after the Korean war), the two countries were polar opposites in terms of their educational and child labor policies. In Korea, child labor was eliminated completely by 1960, and a majority of the resources were being spent on education especially in building a public education system. Koreas educational outcomes escalated positively in terms of enrollment rates, literacy rates, average schooling and was further ahead in terms of development when compared with other countries. Brazil, on the other hand, didn't use its resources wisely. It allocated only a small amount of resources on basic education, and lagged far behind comparable countries in terms of educational outcomes. Child labor laws were not enacted or enforced strictly which resulted in an increase in child labor and such a condition existed well into the 1990s.

If we analyze the existing literature then a majority of researchers agree that child labor bans and regulation should not be used lightly and that there are only exceptional situations and practices, which could work. However, researchers disagree on what exactly these are. Ranjan (1999) in his model shows that any coercive legislation would always be welfare decreasing for some. Many examples from the real world show that child labor regulation only results in more child labor. Coercive child labor laws results in the existent of covert child labor with adverse working and payment conditions. One such instance is observed in Basu and Tzannatos (2003) in the case of India's Child Labor Deterrence Act of 1986. The enactment of such an act resulted in an increase in child labor when firms lowered child labor wages.

Ranjan (2001) argues that child labor legislation may be successful, but the enforcement issue will arise inevitably. The issue of enforcement is also a common problem of estimating the effects of child labor legislation in most child labor models, which assume full compliance.

Recent literature, such as Doepke and Krueger (2006) acknowledge this as a ground for further research. In 2003, two studies came out that criticized child labor regulation, because according to the authors it rarely captured the biggest source of child labor, i.e. domestic industries. Basu and Tzannatos (2003) state that stopping children to work in factories will not have an effect on children working in agriculture. Another study, Bhalotra (2003) comply with the authors in that enforcing minimum wage legislations or trade sanctions have little effect in curbing child labor. Thus, enforcement issues as well as coercive child labor regulations would have little effect on a country moving towards a higher steady-state. Empirical studies and real world situations prove that such coercive laws and enforcement issues have little effect in curbing child labor rather it only results in an increase in the situation with often worse off conditions, thus causing an economy to move towards a lower steady-state than before.

In summary, there are many reasons, for which one must be extremely careful when dealing with restrictive measures against child labor. Coercive measures and direct bans could lead to many pitfalls as aforementioned. Thus many authors suggest that rather than directly banning child labor which could only lead to covert child labor, legislations should be made which target protecting working children and their work conditions, as opposed to removing them from work (Rogers and Swinnerton, 2008). Others suggest policies which focus on motivating parents to send their children to work as opposed to those who coerce them to do so (Ranjan, 2001). Thus, the literature suggests that one should always consider carefully the effect and any alternative policies to reach the intended goal.

# 7 Appendix A

This section gives a detailed illustration of the working to derive the optimal solutions of $e^*, n^*$ and $N^*$ in all three regimes i.e. when (i) $\frac{k}{\phi} \leq h_t \leq \underline{h}_t$ (ii) $\underline{h}_t < h_t < \bar{h}_t$ (iii) $h_t \geq \bar{h}_t$. It also shows how to solve for the values of $\underline{h}_t$ and $\bar{h}_t$ :

The first order conditions of the maximization problem are:

$$\frac{\partial V}{\partial e_{t+1}} = \frac{(\gamma + \beta\alpha_1)}{[h_t(1-\phi n_{t+1}) + kn_{t+1}(1-\phi) - e_{t+1}n_{t+1}(h_t+k)]}[-n_{t+1}(h_t+k)] + \frac{\beta\alpha_2}{(e_{t+1}+\theta)} = 0 \quad \text{(A.1)}$$

$$\frac{\partial V}{\partial n_{t+1}} = \frac{(\gamma + \beta\alpha_1)}{[h_t(1-\phi n_{t+1}) + kn_{t+1}(1-\phi) - e_{t+1}n_{t+1}(h_t+k)]}[-\phi h_t + k(1-\phi) - e_{t+1}(h_t+k)] + \frac{\beta(1-\alpha_1)}{n_{t+1}} = 0 \quad \text{(A.2)}$$

From eq.A.1:

$$\frac{(\gamma+\beta\alpha_1)}{[h_t(1-\phi n_{t+1})+kn_{t+1}(1-\phi)-e_{t+1}n_{t+1}(h_t+k)]}[-n_{t+1}(h_t+k)] = -\frac{\beta\alpha_2}{(e_{t+1}+\theta)}$$

$$n_{t+1} = \frac{\beta\alpha_2[h_t(1-\phi n_{t+1}) + kn_{t+1}(1-\phi) - e_{t+1}n_{t+1}(h_t+k)]}{(e_{t+1}+\theta)(\gamma+\beta\alpha_1)(h_t+k)} \quad \text{(A.3)}$$

where $[h_t(1-\phi n_{t+1}) + kn_{t+1}(1-\phi) - e_{t+1}n_{t+1}(h_t+k)] = c_t$

Substitute eq.A.3 in eq.A.2:

$$\frac{(\gamma+\beta\alpha_1)}{c_t}[-\phi h_t + k(1-\phi) - e_{t+1}(h_t+k)] = \frac{-\beta(1-\alpha_1)(e_{t+1}+\theta)(h_t+k)(\gamma+\beta\alpha_1)}{\beta\alpha_2 c_t}$$

$$\alpha_2[-\phi h_t + k(1-\phi) - e_{t+1}(h_t+k)] = -(1-\alpha_1)(e_{t+1}+\theta)(h_t+k)$$

$$\alpha_2[-\phi h_t + k(1-\phi)] - \alpha_2 e_{t+1}(h_t+k) = -(1-\alpha_1)e_{t+1}(h_t+k) - \theta(1-\alpha_1)(h_t+k)$$

$$e_{t+1}(1-\alpha_1)(h_t+k) - \alpha_2 e_{t+1}(h_t+k) = -\alpha_2[-\phi h_t + k(1-\phi)] - \theta(1-\alpha_1)(h_t+k)$$

$$e_{t+1}\left(h_t+k\right)\left(1-\alpha_1-\alpha_2\right)=\alpha_2\left[\phi h_t-k\left(1-\phi\right)\right]-\theta\left(1-\alpha_1\right)\left(h_t+k\right)$$

$$e_2^*=\frac{\left(h_t+k\right)\left[\alpha_2\phi-\theta\left(1-\alpha_1\right)\right]-k\alpha_2}{\left(h_t+k\right)\left(1-\alpha_1-\alpha_2\right)} \tag{A.4}$$

Eq.A.4 is the optimal value of $e^*$ in the second regime denoted by $e_2^*$ .

From eq.A.3:

$$n_{t+1}=\frac{\beta\alpha_2[h_t(1-\phi n_{t+1})+kn_{t+1}(1-\phi)-e_{t+1}n_{t+1}(h_t+k)]}{(e_{t+1}+\theta)(\gamma+\beta\alpha_1)(h_t+k)}$$

$$n_{t+1}=\frac{\beta\alpha_2h_t+\beta\alpha_2n_{t+1}[k(1-\phi)-e_{t+1}(h_t+k)-\phi h_t]}{e_{t+1}(\gamma+\beta\alpha_1)(h_t+k)+\theta(\gamma+\beta\alpha_1)(h_t+k)}$$

$$n_{t+1}\left[e_{t+1}\left(\gamma+\beta\alpha_1\right)\left(h_t+k\right)+\theta\left(\gamma+\beta\alpha_1\right)\left(h_t+k\right)\right]-$$

$$\beta\alpha_2n_{t+1}\left[k\left(1-\phi\right)-e_{t+1}\left(h_t+k\right)-\phi h_t\right]=\beta\alpha_2h_t$$

$$n_{t+1}\left[e_{t+1}\left(\gamma+\beta\alpha_1\right)\left(h_t+k\right)+\theta\left(\gamma+\beta\alpha_1\right)\left(h_t+k\right)-\beta\alpha_2k\left(1-\phi\right)+\beta\alpha_2\phi h_t+\beta\alpha_2e_{t+1}\left(h_t+k\right)\right]=$$

$$\beta\alpha_2h_t$$

$$n_{t+1}\left[e_{t+1}\left(\gamma+\beta\alpha_1+\beta\alpha_2\right)\left(h_t+k\right)+\theta\left(\gamma+\beta\alpha_1\right)\left(h_t+k\right)-\beta\alpha_2k+\beta\alpha_2k\phi+\beta\alpha_2\phi h_t\right]=$$

$$\beta\alpha_2h_t$$

$$n_{t+1}=\frac{\beta\alpha_2h_t}{(h_t+k)[e_{t+1}(\gamma+\beta\alpha_1+\beta\alpha_2)+\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2k}$$

$$n_{t+1}=\frac{\beta\alpha_2h_t}{\left(h_t+k\right)\left[\left(e_{t+1}+\theta\right)\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2\left(e_{t+1}+\phi\right)\right]-\beta\alpha_2k} \tag{A.5}$$

Eq.A.5 shows $n_{t+1}$ in terms of $e_{t+1}$.

Substituting $e_1^*=0$ in eq.A.5:

$$n_1^*=\frac{\beta\alpha_2h_t}{\left(h_t+k\right)\left[\theta\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2\phi\right]-\beta\alpha_2k} \tag{A.6}$$

Eq.6 shows the value of $n^*$ in the first regime when $e^*=0$ denoted as $n_1^*$.

Substitute eq.A.4 in eq.A.5:

$$n_{t+1}=\frac{\beta\alpha_2h_t}{(h_t+k)\left\{\frac{[\phi h_t-k(1-\phi)]\alpha_2-\theta(1-\alpha_1)(h_t+k)}{(h_t+k)(1-\alpha_1-\alpha_2)}+\theta\right\}(\gamma+\beta\alpha_1)+\beta\alpha_2\left\{\frac{[\phi h_t-k(1-\phi)]\alpha_2-\theta(1-\alpha_1)(h_t+k)}{(h_t+k)(1-\alpha_1-\alpha_2)}+\phi\right\}-\beta\alpha_2k}$$

$$n_{t+1}=\frac{\beta\alpha_2h_t}{(h_t+k)\left\{\left[\frac{(h_t+k)(\alpha_2\phi-\theta+\theta\alpha_1)-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}\right](\gamma+\beta\alpha_1+\beta\alpha_2)+\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi\right\}-\beta\alpha_2k}$$

$$n_{t+1} = \frac{\beta\alpha_2 h_t}{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]+\left[\frac{(h_t+k)(\alpha_2\phi-\theta+\theta\alpha_1)-k\alpha_2}{(1-\alpha_1-\alpha_2)}\right](\gamma+\beta\alpha_1+\beta\alpha_2)-\beta\alpha_2 k}$$

$$n_{t+1} = \frac{\beta\alpha_2 h_t (1-\alpha_1-\alpha_2)}{\begin{array}{c}(h_t+k)(1-\alpha_1-\alpha_2)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi] \\ +[(h_t+k)(\alpha_2\phi-\theta+\theta\alpha_1)-k\alpha_2](\gamma+\beta\alpha_1+\beta\alpha_2) \\ -\beta\alpha_2 k(1-\alpha_1-\alpha_2)\end{array}} \tag{A.7}$$

Solving the denominator of eq.A.7:

$$\begin{aligned}
&=\theta(\gamma+\beta\alpha_1)(h_t+k)(1-\alpha_1-\alpha_2)+(h_t+k)\beta\alpha_2\phi(1-\alpha_1-\alpha_2) \\
&+[(h_t+k)(\alpha_2\phi-\theta+\theta\alpha_1)](\gamma+\beta\alpha_1+\beta\alpha_2)-k\alpha_2(\gamma+\beta\alpha_1)-\beta\alpha_2 k(1-\alpha_1) \\
&=(h_t+k)\left\{\begin{array}{c}\theta(\gamma+\beta\alpha_1)(1-\alpha_1-\alpha_2)+\beta\alpha_2\phi(1-\alpha_1-\alpha_2) \\ +[(\alpha_2\phi-\theta(1-\alpha_1))](\gamma+\beta\alpha_1+\beta\alpha_2)\end{array}\right\} \\
&-k\alpha_2(\gamma+\beta\alpha_1)-\beta\alpha_2 k(1-\alpha_1) \\
&=(h_t+k)\left\{\begin{array}{c}\theta(\gamma+\beta\alpha_1)(1-\alpha_1)-\theta\alpha_2(\gamma+\beta\alpha_1)+\beta\alpha_2\phi(1-\alpha_1) \\ -\beta\alpha_2^2\phi+\alpha_2\phi(\gamma+\beta\alpha_1)+\beta\alpha_2^2\phi-\theta(\gamma+\beta\alpha_1)(1-\alpha_1)-\theta\beta\alpha_2(1-\alpha_1)\end{array}\right\} \\
&-\alpha_2 k(\gamma+\beta\alpha_1)-\beta\alpha_2 k(1-\alpha_1) \\
&=(h_t+k)\{\alpha_2(\gamma+\beta\alpha_1)(\phi-\theta)+\beta\alpha_2(1-\alpha_1)(\phi-\theta)\} \\
&-\alpha_2 k(\gamma+\beta\alpha_1)-\beta\alpha_2 k(1-\alpha_1)
\end{aligned}$$

Substituting the simplified denominator in eq.A.7:

$$n_{t+1} = \frac{\beta\alpha_2 h_t(1-\alpha_1-\alpha_2)}{(h_t+k)\{\alpha_2(\gamma+\beta\alpha_1)(\phi-\theta)+\beta\alpha_2(1-\alpha_1)(\phi-\theta)\}-\alpha_2 k(\gamma+\beta\alpha_1)-\beta\alpha_2 k(1-\alpha_1)}$$

$$n_{t+1} = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(h_t+k)\{(\gamma+\beta\alpha_1)(\phi-\theta)+\beta(1-\alpha_1)(\phi-\theta)\}-k(\gamma+\beta\alpha_1)-\beta k(1-\alpha_1)}$$

$$n_{t+1} = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(h_t+k)(\gamma+\beta\alpha_1+\beta-\beta\alpha_1)(\phi-\theta)-k(\gamma+\beta\alpha_1+\beta-\beta\alpha_1)}$$

$$n_{t+1} = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(h_t+k)(\gamma+\beta)(\phi-\theta)-k(\gamma+\beta)}$$

$$n_2^* = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]} \tag{A.8}$$

Eq.A.8 shows the value of $n^*$ in the second regime denoted as $n_2^*$.

Substituting $e_3^* = (1-\phi)$ in eq.A.5:

$$n_{t+1} = \frac{\beta\alpha_2 h_t}{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2(1-\phi+\phi)]-\beta\alpha_2 k}$$

$$n_3^* = \frac{\beta\alpha_2 h_t}{(h_t+k)\left[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2\right]-\beta\alpha_2 k} \tag{A.9}$$

Eq.9 shows the value of $n^*$ in the third regime denoted by $n_3^*$.

In order to solve for $N^*$ :

$$N_t = \frac{c_t}{n_{t+1}} = \frac{h_t(1-\phi n_{t+1})+k(1-e_{t+1}-\phi)n_{t+1}-e_{t+1}n_{t+1}h_t}{n_{t+1}}$$

Simplifying:

$$N_t = \frac{h_t}{n_{t+1}} + \left[k(1-\phi) - \phi h_t - e_{t+1}(h_t+k)\right]$$

$$N_t = \frac{h_t}{n_{t+1}} + \left[k - (h_t+k)(e_{t+1}+\phi)\right] \tag{A.10}$$

Substitute $e_1^* = 0$ and eq.A.6 in eq.A.10:

$$N_t = \frac{h_t[(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k]}{\beta\alpha_2 h_t} + \left[k-(h_t+k)\phi\right]$$
$$N_t = \frac{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k+\beta\alpha_2[k-(h_t+k)\phi]}{\beta\alpha_2}$$

$$N_1^* = \frac{(h_t+k)\,\theta\,(\gamma+\beta\alpha_1)}{\beta\alpha_2} \tag{A.11}$$

Eq.A.11 is the optimal value of $N^*$ in the first regime denoted as $N_1^*$.

Substitute eq.A.4 and eq.A.8 in eq.A.10 to solve for $N^*$ in the second regime :

$$N_t = h_t\frac{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}{\beta h_t(1-\alpha_1-\alpha_2)} + \left[k-(h_t+k)\left(\frac{(h_t+k)(\alpha_2\phi-\theta(1-\alpha_1))-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}+\phi\right)\right]$$
$$N_t = \frac{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}{\beta(1-\alpha_1-\alpha_2)} + \left\{k-(h_t+k)\left[\frac{(h_t+k)(\alpha_2\phi-\theta(1-\alpha_1))-k\alpha_2+\phi(h_t+k)(1-\alpha_1-\alpha_2)}{(h_t+k)(1-\alpha_1-\alpha_2)}\right]\right\}$$
$$N_t = \frac{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}{\beta(1-\alpha_1-\alpha_2)} + \frac{k(1-\alpha_1-\alpha_2)-(h_t+k)(\alpha_2\phi-\theta(1-\alpha_1))+k\alpha_2-\phi(h_t+k)(1-\alpha_1-\alpha_2)}{(1-\alpha_1-\alpha_2)}$$

$$N_t = \frac{\begin{array}{c}(\gamma+\beta)\left[(h_t+k)(\phi-\theta)-k\right]+\beta k\left(1-\alpha_1-\alpha_2\right) \\ -\beta\left(h_t+k\right)\left(\alpha_2\phi-\theta\left(1-\alpha_1\right)\right)+\beta k\alpha_2 \\ -\beta\phi\left(h_t+k\right)\left(1-\alpha_1-\alpha_2\right)\end{array}}{\beta\left(1-\alpha_1-\alpha_2\right)} \tag{A.12}$$

Simplifying the numerator of eq.A.12:

$$=(h_t+k)\left\{(\gamma+\beta)(\phi-\theta)-\beta\phi\alpha_2+\beta\theta\left(1-\alpha_1\right)-\beta\phi\left(1-\alpha_1-\alpha_2\right)\right\}-(\gamma+\beta)k+\beta k\left(1-\alpha_1\right)$$

$$=(h_t+k)\left\{\gamma(\phi-\theta)-\beta\theta\alpha_1+\beta\phi\alpha_1\right\}+k\left(\beta-\beta\alpha_1-\gamma-\beta\right)$$

$$=(h_t+k)\left[\gamma(\phi-\theta)+\beta\alpha_1(\phi-\theta)\right]-k\left(\gamma+\beta\alpha_1\right)$$

$$=(h_t+k)(\phi-\theta)\left(\gamma+\beta\alpha_1\right)-k\left(\gamma+\beta\alpha_1\right)$$

$$=\left(\gamma+\beta\alpha_1\right)\left[(h_t+k)(\phi-\theta)-k\right]$$

Plugging the simplified numerator in eq.A.12:

$$N_2^* = \frac{\left(\gamma+\beta\alpha_1\right)\left[(h_t+k)(\phi-\theta)-k\right]}{\beta\left(1-\alpha_1-\alpha_2\right)} \tag{A.13}$$

Equation A.13 is the optimal value of $N^*$ in the second regime denoted as $N_2^*$ .

Substituting $e_3^* = (1-\phi)$ and eq.9 to solve for $N^*$in the third regime:

$$N_t = h_t\frac{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}{\beta\alpha_2 h_t}+\left[k-(h_t+k)(1-\phi+\phi)\right]$$

$$N_t = \frac{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k-\beta\alpha_2 h_t}{\beta\alpha_2}$$

$$N_t = \frac{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2(h_t+k)}{\beta\alpha_2}$$

$$N_t = \frac{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2-\beta\alpha_2]}{\beta\alpha_2}$$

$$N_3^* = \frac{(h_t+k)(1-\phi+\theta)\left(\gamma+\beta\alpha_1\right)}{\beta\alpha_2} \tag{A.14}$$

Eq.A.14 gives the value of $N_3^*$ in the third regime.

To find $\underline{h}_t$, substitute $e^* = 0$ in eq.A.4:

$$e^* = \frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}$$

$$0 = \frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}$$

$$0 = (h_t + k)\left[\alpha_2\phi - \theta\left(1-\alpha_1\right)\right] - k\alpha_2$$

$$k\alpha_2 = (h_t + k)\left[\alpha_2\phi - \theta\left(1-\alpha_1\right)\right]$$

$$k\alpha_2 = (h_t + k)\left(\alpha_2\phi - \theta + \theta\alpha_1\right)$$

$$k\alpha_2 - k\left[\alpha_2\phi - \theta\left(1-\alpha_1\right)\right] = h_t\left[\alpha_2\phi - \theta\left(1-\alpha_1\right)\right]$$

$$h_t = \frac{k\alpha_2-k[\alpha_2\phi-\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}$$

$$h_t = \frac{k[\alpha_2-\alpha_2\phi+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}$$

$$h_t = \frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)} \approx \underline{h}_t$$

$$\underline{h}_t = \frac{k\left[\alpha_2\left(1-\phi\right)+\theta\left(1-\alpha_1\right)\right]}{\alpha_2\phi-\theta\left(1-\alpha_1\right)} \tag{A.15}$$

To find $\bar{h}_t$,substitute $e^* = (1-\phi)$ in eq.A.4:

$$e^* = \frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}$$

$$(1-\phi) = \frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}$$

$$(1-\phi)\left(h_t+k\right)\left(1-\alpha_1-\alpha_2\right) = (h_t+k)\left[\alpha_2\phi-\theta\left(1-\alpha_1\right)\right]-k\alpha_2$$

$$h_t(1-\phi)\left(1-\alpha_1-\alpha_2\right) + k(1-\phi)\left(1-\alpha_1-\alpha_2\right) =$$

$$h_t\left[\alpha_2\phi-\theta\left(1-\alpha_1\right)\right] + k\left[\alpha_2\phi-\theta\left(1-\alpha_1\right)\right] - k\alpha_2$$

$$h_t(1-\phi)\left(1-\alpha_1\right) - h_t\alpha_2(1-\phi) + k(1-\phi)\left(1-\alpha_1\right) - k\alpha_2(1-\phi) =$$

$$h_t\alpha_2\phi - \theta h_t\left(1-\alpha_1\right) + k\alpha_2\phi - k\theta\left(1-\alpha_1\right) - k\alpha_2$$

$$h_t(1-\phi)\left(1-\alpha_1\right) - h_t\alpha_2 + k(1-\phi)\left(1-\alpha_1\right) - k\alpha_2 = -\theta h_t\left(1-\alpha_1\right) - k\theta\left(1-\alpha_1\right) - k\alpha_2$$

$$h_t(1-\phi)\left(1-\alpha_1\right) - h_t\alpha_2 + k(1-\phi)\left(1-\alpha_1\right) = -\theta h_t\left(1-\alpha_1\right) - k\theta\left(1-\alpha_1\right)$$

$$h_t(1-\phi)\left(1-\alpha_1\right) - h_t\alpha_2 + \theta h_t\left(1-\alpha_1\right) = -k\theta\left(1-\alpha_1\right) - k(1-\phi)\left(1-\alpha_1\right)$$

$$-h_t\left[\alpha_2 - \theta\left(1-\alpha_1\right) - \left(1-\phi\right)\left(1-\alpha_1\right)\right] = -k\left(1-\alpha_1\right)\left(1+\theta-\phi\right)$$

$$h_t\left[\alpha_2 - \left(1-\alpha_1\right)\left(1+\theta-\phi\right)\right] = k\left(1-\alpha_1\right)\left(1+\theta-\phi\right)$$

$$h_t = \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]} \approx \bar{h}_t$$

$$\bar{h}_t = \frac{k\left(1-\alpha_1\right)\left(1+\theta-\phi\right)}{\left[\alpha_2-\left(1-\alpha_1\right)\left(1+\theta-\phi\right)\right]} \tag{A.16}$$

# 8 Appendix B

This Appendix shows a detailed proof to ensure that $\underline{h}_t < \bar{h}_t$ and that $h_t$ remains positive in the first regime.

In order to ensure that human capital remains positive in the first regime it is assumed that child labor wage cannot exceed the child rearing cost i.e.

$$k \leq \phi w_t h_t$$

where $w_t = 1$,hence

$$k \leq \phi h_t, \text{ thus at } t = 0$$

$$k \leq \phi h_0$$

$$\frac{k}{\phi} \leq \phi h_0$$

i.e.

$$h_0 \geq \frac{k}{\phi} \tag{B.1}$$

Thus the inequality $h_0 \geq \frac{k}{\phi}$ satisfies the condition for human capital to remain positive in the first regime.

To ensure that $\underline{h}_t < \bar{h}_t$ :

$$\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)} < \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

$$k\left[\frac{\alpha_2}{\alpha_2\phi-\theta(1-\alpha_1)} - 1\right] < \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

Dividing by k on both sides:

$$\left[\frac{\alpha_2}{\alpha_2\phi-\theta(1-\alpha_1)} - 1\right] < \frac{(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

$$\frac{\alpha_2^2-\alpha_2(1-\alpha_1)(1+\theta-\phi)}{\alpha_2\phi-\theta(1-\alpha_1)} - [\alpha_2 - (1-\alpha_1)(1+\theta-\phi)] < (1-\alpha_1)(1+\theta-\phi)$$

$$\frac{\alpha_2[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}{\alpha_2\phi-\theta(1-\alpha_1)} < (1-\alpha_1)(1+\theta-\phi) + [\alpha_2 - (1-\alpha_1)(1+\theta-\phi)]$$

$$\alpha_2[\alpha_2 - (1-\alpha_1)(1+\theta-\phi)] < \alpha_2[\alpha_2\phi - \theta(1-\alpha_1)]$$

$$\alpha_2 < \alpha_2\phi - \theta(1-\alpha_1) + (1-\alpha_1)(1-\phi) + \theta(1-\alpha_1)$$

$$\alpha_2 < \alpha_2\phi + (1-\alpha_1) - \phi + \phi\alpha_1$$

$$\alpha_2 < 1 - \alpha_1 - \phi(1-\alpha_1-\alpha_2)$$

$$\phi\,(1-\alpha_1-\alpha_2) < (1-\alpha_1-\alpha_2)$$

Dividing $(1-\alpha_1-\alpha_2)$ on both sides:

$$\phi < 1 \tag{B.2}$$

Thus, the condition $\phi < 1$ holds true and ensures that $\underline{h}_t < \bar{h}_t$.

# 9 Appendix C

This appendix gives a detailed illustration of the working to show the graphical representation of $e^*, n^*$ and $N^*$ in the following three regimes:

(i) $\frac{k}{\phi} \leq h_t \leq \underline{h}_t$

(ii) $\underline{h}_t < h_t < \bar{h}_t$

(iii) $h_t \geq \bar{h}_t$

**Time dedicated to Education:**

(i) In this regime i.e. when $h_t$ is in the interval $\left[\frac{k}{\phi}, \underline{h}_t\right]$, the time dedicated to education is a constant value of $e_{t+1} = 0$, hence the straight line overlapping the x-axis.

(ii) In this regime i.e. when $h_t$ is in the interval $\left(\underline{h}_t, \bar{h}_t\right)$, the way $e^*$ behaves can be determined by taking the first order condition of the optimal value of $e_2^*$ with respect to $h_t$:

$$e_2^* = \frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}$$

$$\frac{\partial e_2^*}{\partial h_t} = \frac{\phi\alpha_2(h_t+k)(1-\alpha_1-\alpha_2)-\{[\phi h_t-k(1-\phi)]\alpha_2\}(1-\alpha_1-\alpha_2)}{[(h_t+k)(1-\alpha_1-\alpha_2)]^2}$$

Ensuring that $\frac{\partial e_2^*}{\partial h_t} > 0$

$$\frac{\partial e_2^*}{\partial h_t} = \frac{\phi\alpha_2(h_t+k)(1-\alpha_1-\alpha_2)-\{[\phi h_t-k(1-\phi)]\alpha_2\}(1-\alpha_1-\alpha_2)}{[(h_t+k)(1-\alpha_1-\alpha_2)]^2} > 0$$

$\frac{\partial e_2^*}{\partial h_t} = \frac{\phi\alpha_2 k+k(1-\phi)\alpha_2}{(h_t+k)^2(1-\alpha_1-\alpha_2)} > 0$, since $(1-\alpha_1-\alpha_2) > 0$ and condition B.2 holds true.

$$\frac{\partial e_2^*}{\partial h_t} = \frac{k\alpha_2}{(h_t+k)^2(1-\alpha_1-\alpha_2)} > 0 \tag{C.1}$$

Taking the second order conditions:

$$\frac{\partial^2 e_2^*}{(\partial h_t)^2} = -\frac{2k\alpha_2}{(h_t+k)^3(1-\alpha_1-\alpha_2)}$$

Determining if $\frac{\partial^2 e_2^*}{(\partial h_t)^2} \lesseqgtr 0$ :

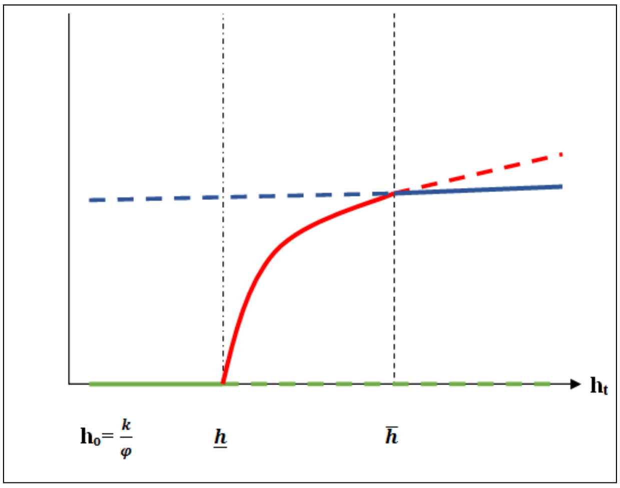
$\frac{\partial^2 e_2^*}{(\partial h_t)^2} - \frac{2k\alpha_2}{(h_t+k)^3(1-\alpha_1-\alpha_2)} < 0$, since $(1-\alpha_1-\alpha_2) > 0$.

$$\frac{\partial^2 e_2^*}{(\partial h_t)^2} = -\frac{2k\alpha_2}{(h_t+k)^3(1-\alpha_1-\alpha_2)} < 0 \tag{C.2}$$

This results in an increasing concave shaped graph.

(iii) In the third regime i.e. when $h_t$ is in the interval $\left[\bar{h}_t, \infty\right)$ , the time dedicated to education has a constant value of $(1-\phi)$ , hence a straight line.

The analysis yields the following graphical representation of $e^*$.In this graph and the following graphs, the different colors show the behavior of the variable in different regimes. Regime 1 is shown by green color, Regime 2 is shown by red color and Regime 3 is shown by blue color.




**Child Nutrition:**

(i) In this regime, take the foc of eq.A.11 with respect to $h_t$ :

$$N_1^* = \frac{(h_t+k)\theta(\gamma+\beta\alpha_1)}{\beta\alpha_2}$$
$$\frac{\partial N_1^*}{\partial h_t} = \frac{\theta(\gamma+\beta\alpha_1)}{\beta\alpha_2}$$

Since all parameters are positive, hence:

$$\frac{\partial N_1^*}{\partial h_t} = \frac{\theta\left(\gamma+\beta\alpha_1\right)}{\beta\alpha_2} > 0. \tag{C.3}$$

(ii) In the second regime, take the foc of eq.A.13:

$$N_2^* = \frac{(\gamma+\beta\alpha_1)[(h_t+k)(\phi-\theta)-k]}{\beta(1-\alpha_1-\alpha_2)}$$
$$\frac{\partial N_2^*}{\partial h_t} = \frac{(\gamma+\beta\alpha_1)(\phi-\theta)}{\beta(1-\alpha_1-\alpha_2)}$$
$$\text{Ensuring that } \frac{\partial N_2^*}{\partial h_t} > 0$$
$$\frac{(\gamma+\beta\alpha_1)(\phi-\theta)}{\beta(1-\alpha_1-\alpha_2)} > 0$$

$$(\gamma + \beta\alpha_1)(\phi - \theta) > 0$$
$$(\phi - \theta) > 0$$
$$\phi > \theta$$
or

$$\theta < \phi \tag{C.4}$$

Thus, since C.4 holds true hence:

$$\frac{\partial N_2^*}{\partial h_t} = \frac{(\gamma + \beta\alpha_1)(\phi - \theta)}{\beta(1 - \alpha_1 - \alpha_2)} > 0 \tag{C.5}$$

(iii) In the third regime, take the foc of eq.A.14:

$$N_3^* = \frac{(h_t+k)(1-\phi+\theta)(\gamma+\beta\alpha_1)}{\beta\alpha_2}$$
$$\frac{\partial N_3^*}{\partial h_t} = \frac{(\gamma+\beta\alpha_1)(1-\phi+\theta)}{\beta\alpha_2}$$
Ensuring that $\frac{\partial N_3^*}{\partial h_t} > 0$
$$\frac{(\gamma+\beta\alpha_1)(1-\phi+\theta)}{\beta\alpha_2} > 0$$

$$(1 - \phi + \theta) > 0 \tag{C.6}$$

Since, C.6 holds true hence,

$$\frac{\partial N_3^*}{\partial h_t} = \frac{(\gamma + \beta\alpha_1)(1 - \phi + \theta)}{\beta\alpha_2} > 0 \tag{C.7}$$

However, since Child nutrition is an increasing function in all three regimes, it remains to be seen as to which slope is larger compared to the other.

To determine this first compare the slope of Child nutrition in the first regime with that of the third regime:

Slope of Child nutrition in first regime < Slope of Child nutrition in third regime

$$\frac{\theta(\gamma+\beta\alpha_1)}{\beta\alpha_2} < \frac{(\gamma+\beta\alpha_1)(\theta-\phi+1)}{\beta\alpha_2}$$

Dividing by $(\gamma + \beta\alpha_1)$ and multiplying by $\beta\alpha_2$ on both sides:

$$\theta < \theta - \phi + 1$$

Simplifying gives the following:

$$\phi < 1 \tag{C.8}$$

Thus, the slope of first regime is less than slope of third regime since C.8 holds true.

Now compare the slope of the second and third regime:

Slope of Child nutrition in second regime > Slope of Child nutrition in third regime

$$\frac{(\gamma+\beta\alpha_1)(\phi-\theta)}{\beta(1-\alpha_1-\alpha_2)} > \frac{(\gamma+\beta\alpha_1)(\theta-\phi+1)}{\beta\alpha_2}$$

Dividing by $(\gamma + \beta\alpha_1)$ and multiplying by $\beta$ on both sides:

$$\frac{(\phi-\theta)}{(1-\alpha_1-\alpha_2)} > \frac{(\theta-\phi+1)}{\alpha_2}$$

$$(\phi - \theta)\,\alpha_2 > (\theta - \phi + 1)\,(1 - \alpha_1 - \alpha_2)$$

$$0 > \theta\,(1 - \alpha_1) - \phi\,(1 - \alpha_1) + (1 - \alpha_1 - \alpha_2)$$

$$\phi\,(1 - \alpha_1) - \theta\,(1 - \alpha_1) > (1 - \alpha_1 - \alpha_2)$$

$$(\phi - \theta)\,(1 - \alpha_1) > (1 - \alpha_1 - \alpha_2)$$

$$(\phi - \theta) > \frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)}$$

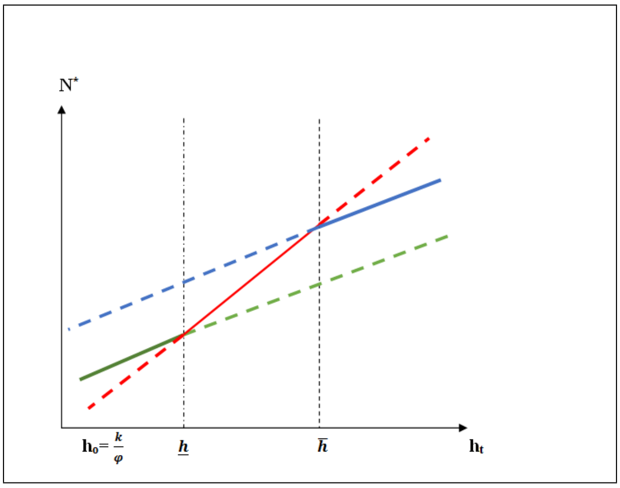
$$\phi - \frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)} > \theta$$

or

$$\theta < \phi - \frac{(1 - \alpha_1 - \alpha_2)}{(1 - \alpha_1)} \tag{C.9}$$

Thus, slope of second regime is greater than slope of third regime when C.9 holds true.

The analysis yields the following graphical representation of $N^*$ in all three regimes:




**Number of Children/Fertility:**

(i) In this regime, take the foc of eq.A.6 with respect to $h_t$ :

$$n_1^* = \frac{\beta\alpha_2 h_t}{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$

$$\frac{\partial n_1^*}{\partial h_t} = \frac{\beta\alpha_2\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}-\beta\alpha_2 h_t[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^2}$$

$$\frac{\partial n_1^*}{\partial h_t} = \frac{\beta\alpha_2 k[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi-\beta\alpha_2]}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^2}$$

$$\frac{\partial n_1^*}{\partial h_t} = \frac{\beta\alpha_2 k[\theta(\gamma+\beta\alpha_1)-\beta\alpha_2(1-\phi)]}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^2}$$

$$\frac{\partial n_1^*}{\partial h_t} = \frac{k\left[\beta\alpha_2\theta\left(\gamma+\beta\alpha_1\right)-\beta^2\alpha_2^2\left(1-\phi\right)\right]}{\{(h_t+k)\left[\theta\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2\phi\right]-\beta\alpha_2 k\}^2} \tag{C.10}$$

Ensuring that $\frac{\partial n_1^*}{\partial h_t} > 0$

$$\frac{k\left[\beta\alpha_2\theta(\gamma+\beta\alpha_1)-\beta^2\alpha_2^2(1-\phi)\right]}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^2} > 0$$

$$\left[\beta\alpha_2\theta\left(\gamma+\beta\alpha_1\right)-\beta^2\alpha_2^2\left(1-\phi\right)\right] > 0$$

$$\beta\alpha_2\theta\left(\gamma+\beta\alpha_1\right) > \beta^2\alpha_2^2\left(1-\phi\right)$$

$$\theta\left(\gamma+\beta\alpha_1\right) > \beta\alpha_2\left(1-\phi\right)$$

$$\theta > \frac{\beta\alpha_2\left(1-\phi\right)}{\left(\gamma+\beta\alpha_1\right)} \tag{C.11}$$

Thus, when C.11 holds, $\frac{\partial n_1^*}{\partial h_t} > 0$ in first regime.

Taking the soc:

$$\frac{\partial^2 n_1^*}{(\partial h_t)^2} = -2\frac{[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]\{\beta\alpha_2 k[\theta(\gamma+\beta\alpha_1)-\beta\alpha_2+\beta\alpha_2\phi]\}}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^3}$$
$$\frac{\partial^2 n_1^*}{(\partial h_t)^2} = -2\beta\alpha_2 k\frac{\{[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi][\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi-\beta\alpha_2]\}}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^3}$$

$$\frac{\partial^2 n_1^*}{(\partial h_t)^2} = -2\beta\alpha_2 k\frac{\left\{[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]^2-\beta\alpha_2[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]\right\}}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^3} \tag{C.12}$$

In C.12, observing whether the numerator is $\gtreqless 0$

$$[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]^2-\beta\alpha_2[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi] \gtreqless 0$$
$$[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi][\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi-\beta\alpha_2] \gtreqless$$
$$[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi][\theta(\gamma+\beta\alpha_1)-\beta\alpha_2(1-\phi)] \gtreqless 0$$
$$[\theta(\gamma+\beta\alpha_1)-\beta\alpha_2(1-\phi)] \gtreqless 0$$
$$\theta(\gamma+\beta\alpha_1) \gtreqless \beta\alpha_2(1-\phi)$$
$$\theta > \frac{\beta\alpha_2(1-\phi)}{(\gamma+\beta\alpha_1)}$$

The denominator of C.12 can be expanded as :

$$(h_t+k)\,\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi h_t+\beta\alpha_2\phi k-\beta\alpha_2 k$$
$$(h_t+k)\,\theta(\gamma+\beta\alpha_1)+\beta\alpha_2[\phi h_t+k\phi-k]$$

where according to assumption (2.2.5) $k < \phi h_t$, hence $\phi h_t - k > 0$. Thus $(\phi h_t - k + k\phi) > 0$. Since all other parameters are positive, so the cubed denominator $\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi] - \beta\alpha$
0. However, the negative sign with the whole fraction results in the function being $< 0$.

Thus,

$$\frac{\partial^2 n_1^*}{(\partial h_t)^2} = -2\beta\alpha_2 k\frac{\left\{[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]^2-\beta\alpha_2[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]\right\}}{\{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\}^3} < 0 \tag{C.13}$$

In the first regime, $\frac{\partial n_1^*}{\partial h_t} > 0$ and $\frac{\partial^2 n_1^*}{(\partial h_t)^2} < 0$, hence the function is increasing at a decreasing rate. Thus the function is increasing and concave down.

(ii) In this regime take the foc of Eq.A.8:

$$n_2^* = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}$$

$$\frac{\partial n_2^*}{\partial h_t} = \frac{\beta(1-\alpha_1-\alpha_2)\{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]\}-\beta h_t(1-\alpha_1-\alpha_2)(\gamma+\beta)(\phi-\theta)}{\{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]\}^2}$$

$$\frac{\partial n_2^*}{\partial h_t} = \frac{\beta(1-\alpha_1-\alpha_2)(\gamma+\beta)[k(\phi-\theta)-k]}{\{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]\}^2}$$

$$\frac{\partial n_2^*}{\partial h_t} = \frac{\beta(1-\alpha_1-\alpha_2)(\gamma+\beta)k(\phi-\theta-1)}{\{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]\}^2}$$

$$\frac{\partial n_2^*}{\partial h_t} = -\frac{\beta k(1-\alpha_1-\alpha_2)(\gamma+\beta)(1-\phi+\theta)}{\{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]\}^2}$$

Since, it has been previously determined that $(1-\phi+\theta) > 0$, thus

$$\frac{\partial n_2^*}{\partial h_t} = -\frac{\beta k\,(1-\alpha_1-\alpha_2)\,(\gamma+\beta)\,(1-\phi+\theta)}{\{(\gamma+\beta)\,[(h_t+k)\,(\phi-\theta)-k]\}^2} < 0 \tag{C.14}$$

Taking the soc:

$$\frac{\partial^2 n_2^*}{(\partial h_t)^2} =$$

$$-2\,\{(\gamma+\beta)\,[(h_t+k)\,(\phi-\theta)-k]\}^{-3}\,(\gamma+\beta)\,(\phi-\theta)\,\{\beta\,(1-\alpha_1-\alpha_2)\,(\gamma+\beta)\,k\,(\phi-\theta-1)\}$$

$$\frac{\partial^2 n_2^*}{(\partial h_t)^2} = \frac{-2(\gamma+\beta)(\phi-\theta)\{\beta(1-\alpha_1-\alpha_2)(\gamma+\beta)k(\phi-\theta-1)\}}{\{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]\}^3}$$

$$\frac{\partial^2 n_2^*}{(\partial h_t)^2} = \frac{2\beta k(\gamma+\beta)^2(\phi-\theta)(1-\phi+\theta)(1-\alpha_1-\alpha_2)}{(\gamma+\beta)^3[(h_t+k)(\phi-\theta)-k]^3}$$

$$\frac{\partial^2 n_2^*}{(\partial h_t)^2} = \frac{2\beta k(\phi-\theta)(1-\phi+\theta)(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]^3}$$

In the above function, as has been previously determined (see C.6 and C.4), $(1-\phi+\theta) > 0$ and $\theta < \phi$, hence $(\phi-\theta) > 0$.

Thus resulting in the whole function being $> 0$:

$$\frac{\partial^2 n_2^*}{(\partial h_t)^2} = \frac{2\beta k\,(\phi-\theta)\,(1-\phi+\theta)\,(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\,[(h_t+k)\,(\phi-\theta)-k]^3} > 0 \tag{C.15}$$

In this regime, the foc $< 0$ and soc $> 0$, thus the function is decreasing and concave up.

(iii) In third regime, take the foc of eq.A.9

$$n_3^* = \frac{\beta\alpha_2 h_t}{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}$$

$$\frac{\partial n_3^*}{\partial h_t} = \frac{\beta\alpha_2\{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k\}-\beta\alpha_2 h_t[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]}{\{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k\}^2}$$

$$\frac{\partial n_3^*}{\partial h_t} = \frac{\beta\alpha_2 k[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-k\beta^2\alpha_2^2}{\{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k\}^2}$$

$$\frac{\partial n_3^*}{\partial h_t} = \frac{\beta\alpha_2 k(1-\phi+\theta)(\gamma+\beta\alpha_1)+k\beta^2\alpha_2^2-k\beta^2\alpha_2^2}{\{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k\}^2}$$
$$\frac{\partial n_3^*}{\partial h_t} = \frac{\beta\alpha_2 k(1-\phi+\theta)(\gamma+\beta\alpha_1)}{\{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k\}^2}$$

In this regime, $\frac{\partial n_3^*}{\partial h_t} > 0$, since it has previously been established that $(1-\phi+\theta) > 0$, therefore:

$$\frac{\partial n_3^*}{\partial h_t} = \frac{\beta\alpha_2 k\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)}{\left\{\left(h_t+k\right)\left[\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2\right]-\beta\alpha_2 k\right\}^2} > 0 \tag{C.16}$$

Taking the soc:

$$\frac{\partial^2 n_3^*}{(\partial h_t)^2} = -2\left\{\left(h_t+k\right)\left[\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2\right]-\beta\alpha_2 k\right\}^{-3}$$
$$\left[\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2\right]\left[\beta\alpha_2 k\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)\right]$$
$$\frac{\partial^2 n_3^*}{(\partial h_t)^2} = -2\frac{[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2][\beta\alpha_2 k(1-\phi+\theta)(\gamma+\beta\alpha_1)]}{\{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k\}^3}$$
$$\frac{\partial^2 n_3^*}{(\partial h_t)^2} = -2\frac{[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2][\beta\alpha_2 k(1-\phi+\theta)(\gamma+\beta\alpha_1)]}{\{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)]+\beta\alpha_2 h_t\}^3}$$

In the above function, $(1-\phi+\theta) > o$ and since all other parameters are positive, thus the negative sign with the function shows that overall the function is $< 0$. Thus,

$$\frac{\partial^2 n_3^*}{\left(\partial h_t\right)^2} = -2\frac{\left[\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2\right]\left[\beta\alpha_2 k\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)\right]}{\left\{\left(h_t+k\right)\left[\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)\right]+\beta\alpha_2 h_t\right\}^3} < 0 \tag{C.17}$$

In this regime, the foc $> 0$ and soc $< 0$, thus the function is increasing and concave down.

Thus now we have two inequalities:

a) $\theta > \frac{\beta\alpha_2(1-\phi)}{(\gamma+\beta\alpha_1)} \equiv \theta_A$

b) $\theta < \phi - \frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)} \equiv \theta_B$

Now it remains to be seen as to whether $\theta_A \lesseqgtr \theta_B$:

$$\frac{\beta\alpha_2(1-\phi)}{(\gamma+\beta\alpha_1)} \lesseqgtr \phi - \frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)}$$
$$\beta\alpha_2\left(1-\phi\right) \lesseqgtr \left(\gamma+\beta\alpha_1\right)\phi - \left(\gamma+\beta\alpha_1\right)\frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)}$$
$$\beta\alpha_2 - \beta\alpha_2\phi \lesseqgtr \left(\gamma+\beta\alpha_1\right)\phi - \left(\gamma+\beta\alpha_1\right)\frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)}$$
$$\beta\alpha_2 + \left(\gamma+\beta\alpha_1\right)\frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)} \lesseqgtr \left(\gamma+\beta\alpha_1\right)\phi + \beta\alpha_2\phi$$

$$\beta\alpha_2 + (\gamma+\beta\alpha_1)\,\tfrac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)} \lesseqgtr \phi\,(\gamma+\beta\alpha_1+\beta\alpha_2)$$
$$\tfrac{\beta\alpha_2(1-\alpha_1)+(\gamma+\beta\alpha_1)(1-\alpha_1-\alpha_2)}{(1-\alpha_1)(\gamma+\beta\alpha_1+\beta\alpha_2)} \geq \phi$$

$$\phi \leq \frac{\beta\alpha_2\,(1-\alpha_1)+(\gamma+\beta\alpha_1)\,(1-\alpha_1-\alpha_2)}{(1-\alpha_1)\,(\gamma+\beta\alpha_1+\beta\alpha_2)} \equiv \tilde{\phi} \tag{C.18A}$$

Thus $\theta_A > \theta_B$ when $\phi \leq \tilde{\phi}$. $\theta_A$ is the upper limit for the minimum level of skills inherent in a child without any education whereas $\theta_B$ is the lower limit for the minimum level of skills in a child without receiving any education. Thus, $\theta$ would lie in the following interval when $\phi \leq \tilde{\phi}$:

$$\theta_B \approx \phi - \frac{(1-\alpha_1-\alpha_2)}{(1-\alpha_1)} < \theta < \frac{\beta\alpha_2\,(1-\phi)}{(\gamma+\beta\alpha_1)} \approx \theta_A \tag{C.18B}$$

C.18 ensures that

Hence, Inequality C.18 ensures that $\theta$ and $\phi$ are both $< \tilde{\phi}$.

Now, it remains to be seen that at $\underline{h}_t$, $\bar{h}_t$ and $h_t = h_0 = \frac{k}{\phi}$, which of the three optimal solutions obtained for $n^*$ is $\lesseqgtr$ than each other:

Determing $n_1^*\Big|_{h_t=h_0=\frac{k}{\phi}}$ :

$$n_1^* = \tfrac{\beta\alpha_2 h_t}{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$
Substituting $h_t = h_0 = \frac{k}{\phi}$
$$= \tfrac{\beta\alpha_2\frac{k}{\phi}}{\left(\frac{k}{\phi}+k\right)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$
$$= \tfrac{\beta\alpha_2 k}{(k+k\phi)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k\phi}$$
$$= \tfrac{\beta\alpha_2 k}{(k+k\phi)\theta(\gamma+\beta\alpha_1)+\beta\alpha_2 k\phi^2}$$

$$n_1^*\Big|_{h_t=h_0=\frac{k}{\phi}} = \frac{\beta\alpha_2}{(1+\phi)\,\theta\,(\gamma+\beta\alpha_1)+\beta\alpha_2\phi^2} \tag{C.19}$$

Determing $n_2^*\Big|_{h_t=h_0=\frac{k}{\phi}}$ :

$$n_2^* = \tfrac{\beta h_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}$$
Substituting $h_t = h_0 = \frac{k}{\phi}$

$$
\begin{aligned}
&=\frac{\beta\frac{k}{\phi}(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\left[\left(\frac{k}{\phi}+k\right)(\phi-\theta)-k\right]}\\
&=\frac{\beta k(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(k+k\phi)(\phi-\theta)-k\phi]}\\
&=\frac{\beta(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(1+\phi)(\phi-\theta)-\phi]}\\
&=\frac{\beta(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\left[\phi-\theta+\phi^2-\phi\theta-\phi\right]}
\end{aligned}
$$

$$
n_2^*\Big|_{h_t=h_0=\frac{k}{\phi}} = \frac{\beta\left(1-\alpha_1-\alpha_2\right)}{\left(\gamma+\beta\right)\left[\phi^2-\theta\left(1+\phi\right)\right]} \tag{C.20}
$$

Determing $n_3^*\Big|_{h_t=h_0=\frac{k}{\phi}}$ :

$$
\begin{gathered}
n_3^* = \frac{\beta\alpha_2 h_t}{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}\\
\text{Substituting } h_t = h_0 = \frac{k}{\phi}\\
=\frac{\beta\alpha_2\frac{k}{\phi}}{\left(\frac{k}{\phi}+k\right)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}\\
=\frac{\beta\alpha_2 k}{(k+k\phi)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k\phi}\\
=\frac{\beta\alpha_2}{(1+\phi)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2\phi}
\end{gathered}
$$

$$
n_3^*\Big|_{h_t=h_0=\frac{k}{\phi}} = \frac{\beta\alpha_2}{\left(1+\phi\right)\left(1-\phi+\theta\right)\left(\gamma+\beta\alpha_1\right)+\beta\alpha_2} \tag{C.21}
$$

Determing $n_1^*\big|_{h_t=\underline{h}_t}$ :

$$
\begin{gathered}
n_1^* = \frac{\beta\alpha_2 h_t}{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}\\
\text{Substituting } h_t = \underline{h}_t = \frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}\\
=\frac{\beta\alpha_2\underline{h}_t}{(\underline{h}_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}\\
=\frac{\beta\alpha_2\left\{\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}\right\}}{\left[\left(\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}+k\right)\right][\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}\\
=-\beta\frac{\alpha_2}{\phi\alpha_2+\theta(\alpha_1-1)}\frac{k\theta(\alpha_1-1)+k\alpha_2(\phi-1)}{\theta(\gamma+\beta\alpha_1)\left(k-\frac{1}{\phi\alpha_2+\theta(\alpha_1-1)}(k\theta(\alpha_1-1)+k\alpha_2(\phi-1))\right)+k\beta\alpha_2(\phi-1)-\beta\phi\frac{\alpha_2}{\phi\alpha_2+\theta(\alpha_1-1)}(k\theta(\alpha_1-1)+k\alpha_2(\phi-1))}\\
=\frac{1}{\theta(\beta+\gamma)}\left(\theta\beta+\beta\alpha_2-\theta\beta\alpha_1-\beta\phi\alpha_2\right)
\end{gathered}
$$

$$
n_1^*\big|_{h_t=\underline{h}_t} = \beta\frac{\left[\theta\left(1-\alpha_1\right)+\alpha_2\left(1-\phi\right)\right]}{\theta\left(\beta+\gamma\right)} \tag{C.22}
$$

Determing $n_1^*|_{h_t=\bar{h}_t}$ :

$$\text{Substituting } h_t = \bar{h}_t = \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

$$= \frac{\beta\alpha_2\bar{h}_t}{\left(\bar{h}_t+k\right)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$

$$= \frac{\beta\alpha_2\left\{\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}\right\}}{\left[\left(\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}+k\right)\right][\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$

$$= -k\beta\alpha_2(\alpha_1-1)\frac{\theta-\phi+1}{\left(\theta(\gamma+\beta\alpha_1)\left(k-k(\alpha_1-1)\frac{\theta-\phi+1}{\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1}\right)+k\beta\alpha_2(\phi-1)-k\beta\phi\alpha_2(\alpha_1-1)\cdots\right.}$$

$$(\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1)$$

$$= -\beta(\alpha_1-1)\frac{\theta-\phi+1}{\beta+\theta\beta+\theta\gamma-\beta\phi-\beta\alpha_1-\beta\alpha_2+\beta\phi\alpha_1+\beta\phi\alpha_2}$$

$$= \beta(1-\alpha_1)\frac{\theta-\phi+1}{\beta+\theta\beta+\theta\gamma-\beta\phi-\beta\alpha_1-\beta\alpha_2+\beta\phi\alpha_1+\beta\phi\alpha_2}$$

$$= \beta(1-\alpha_1)\frac{\theta-\phi+1}{\beta(1-\alpha_1-\alpha_2)+\theta(\gamma+\beta)-\beta\phi(1-\alpha_1-\alpha_2)}$$

$$n_1^*|_{h_t=\bar{h}_t} = \beta(1-\alpha_1)\frac{\theta-\phi+1}{\beta(1-\alpha_1-\alpha_2)(1-\phi)+\theta(\gamma+\beta)} \quad \text{(C.23)}$$

Determing $n_2^*\big|_{h_t=\underline{h}_t}$ :

$$n_2^* = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}$$

$$\text{Substituting } h_t = \underline{h}_t = \frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)} :$$

$$= \frac{\beta\underline{h}_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(\underline{h}_t+k)(\phi-\theta)-k]}$$

$$= \frac{\beta(1-\alpha_1-\alpha_2)\left\{\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}\right\}}{(\gamma+\beta)\left[\left[\left(\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}+k\right)\right](\phi-\theta)-k\right]}$$

$$= -\frac{\beta}{\phi\alpha_2+\theta(\alpha_1-1)}\frac{k\theta(\alpha_1-1)+k\alpha_2(\phi-1)}{\theta(\beta+\gamma)\left(k-\frac{1}{\phi\alpha_2+\theta(\alpha_1-1)}(k\theta(\alpha_1-1)+k\alpha_2(\phi-1))\right)-k(\phi-1)(\beta+\gamma)+\frac{\phi}{\phi\alpha_2+\theta(\alpha_1-1)}(\beta+\gamma)(k\theta(\alpha_1-1)+k\alpha_2(\phi-1))}$$

$$(\alpha_1+\alpha_2-1)$$

$$= -\frac{\beta}{\phi\alpha_2-\theta(1-\alpha_1)}\frac{-k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)\left(k+\frac{k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\phi\alpha_2-\theta(1-\alpha_1)}\right)+k(1-\phi)(\beta+\gamma)-\frac{\phi}{\phi\alpha_2-\theta(1-\alpha_1)}(\beta+\gamma)k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}$$

$$(\alpha_1+\alpha_2-1)$$

$$= -\frac{\beta}{\phi\alpha_2-\theta(1-\alpha_1)}\frac{-k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)\left(\frac{k[\phi\alpha_2-\theta(1-\alpha_1)]+k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\phi\alpha_2-\theta(1-\alpha_1)}\right)+\frac{k(1-\phi)(\beta+\gamma)[\phi\alpha_2-\theta(1-\alpha_1)]-\phi(\beta+\gamma)k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\phi\alpha_2-\theta(1-\alpha_1)}}(\alpha_1+\alpha_2-1)$$

$$= -\frac{\beta}{\phi\alpha_2-\theta(1-\alpha_1)}\frac{-k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)\left(\frac{k\alpha_2}{\phi\alpha_2-\theta(1-\alpha_1)}\right)+\frac{-k(1-\phi)(\beta+\gamma)\theta(1-\alpha_1)-\phi(\beta+\gamma)k\theta(1-\alpha_1)}{\phi\alpha_2-\theta(1-\alpha_1)}}(\alpha_1+\alpha_2-1)$$

$$= -\frac{\beta}{\phi\alpha_2-\theta(1-\alpha_1)}\frac{-k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)\left(\frac{k\alpha_2}{\phi\alpha_2-\theta(1-\alpha_1)}\right)-\frac{k(\beta+\gamma)\theta(1-\alpha_1)[1-\phi+\phi]}{\phi\alpha_2-\theta(1-\alpha_1)}}(\alpha_1+\alpha_2-1)$$

$$= -\frac{\beta}{\phi\alpha_2-\theta(1-\alpha_1)}\frac{-k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\frac{k\alpha_2\theta(\beta+\gamma)-k\theta(1-\alpha_1)(\beta+\gamma)}{\phi\alpha_2-\theta(1-\alpha_1)}}(\alpha_1+\alpha_2-1)$$

$$= -\frac{\beta}{\phi\alpha_2-\theta(1-\alpha_1)}\frac{-k[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\frac{-k\theta(\beta+\gamma)(1-\alpha_1-\alpha_2)}{\phi\alpha_2-\theta(1-\alpha_1)}}(\alpha_1+\alpha_2-1)$$

$$= -\frac{\beta}{\phi\alpha_2-\theta(1-\alpha_1)}\frac{[\theta(1-\alpha_1)+\alpha_2(1-\phi)][\phi\alpha_2-\theta(1-\alpha_1)]}{\theta(\beta+\gamma)(1-\alpha_1-\alpha_2)}(\alpha_1+\alpha_2-1)$$

$$=-\beta\frac{[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)(1-\alpha_1-\alpha_2)}\left[-\left(1-\alpha_1-\alpha_2\right)\right]$$

$$n_2^*\big|_{h_t=\underline{h}_t}=\beta\frac{\left[\theta\left(1-\alpha_1\right)+\alpha_2\left(1-\phi\right)\right]}{\theta\left(\beta+\gamma\right)} \tag{C.24}$$

Determing $n_2^*|_{h_t=\bar{h}_t}$ :

$$\text{Substituting } h_t=\bar{h}_t=\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

$$=\frac{\beta\bar{h}_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\left[\left(\bar{h}_t+k\right)(\phi-\theta)-k\right]}$$

$$=\frac{\beta\left\{\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}\right\}(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\left[\left(\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}+k\right)(\phi-\theta)-k\right]}$$

$$=-k\beta\frac{\alpha_1-1}{\theta\left(k-k(\alpha_1-1)\frac{\theta-\phi+1}{\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1}\right)(\beta+\gamma)-k(\phi-1)(\beta+\gamma)+k\phi(\alpha_1-1)(\beta+\gamma)\frac{\theta-\phi+1}{\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1}}$$

$$\left(\alpha_1+\alpha_2-1\right)\frac{\theta-\phi+1}{\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1}$$

$$=-\beta\frac{\alpha_1-1}{\beta+\gamma}$$

$$n_2^*|_{h_t=\bar{h}_t}=\beta\frac{1-\alpha_1}{\beta+\gamma} \tag{C.25}$$

Determing $n_3^*\big|_{h_t=\underline{h}_t}$ :

$$n_3^*=\frac{\beta\alpha_2h_t}{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2k}$$

$$\text{Substituting } h_t=\underline{h}_t=\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}$$

$$=\frac{\beta\alpha_2\underline{h}_t}{(\underline{h}_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2k}$$

$$=\frac{\beta\alpha_2\left\{\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}\right\}}{\left[\left(\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}+k\right)\right][(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2k}$$

$$=-\beta\frac{\alpha_2}{(\phi\alpha_2+\theta(\alpha_1-1))\left((\gamma+\beta\alpha_1)\left(k-\frac{1}{\phi\alpha_2+\theta(\alpha_1-1)}(k\theta(\alpha_1-1)+k\alpha_2(\phi-1))\right)(\theta-\phi+1)-\beta\frac{\alpha_2}{\phi\alpha_2+\theta(\alpha_1-1)}(k\theta(\alpha_1-1)+k\alpha_2(\phi-1))\right)}$$

$$\left(k\theta\left(\alpha_1-1\right)+k\alpha_2\left(\phi-1\right)\right)$$

$$=\frac{\theta\beta+\beta\alpha_2-\theta\beta\alpha_1-\beta\phi\alpha_2}{\gamma+\theta\beta+\theta\gamma-\gamma\phi+\beta\alpha_1+\beta\alpha_2-\beta\phi\alpha_1-\beta\phi\alpha_2}$$

$$n_3^*\big|_{h_t=\underline{h}_t}=\frac{\theta\beta\left(1-\alpha_1\right)+\beta\alpha_2\left(1-\phi\right)}{\theta\left(\beta+\gamma\right)+\left(1-\phi\right)\left(\gamma+\beta\alpha_1+\beta\alpha_2\right)} \tag{C.26}$$

Determining $n_3^*|_{h_t=\bar{h}_t}$ :

$$\text{Substituting } h_t = \bar{h}_t = \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

$$= \frac{\beta\alpha_2\bar{h}_t}{\left(\bar{h}_t+k\right)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}$$

$$= \frac{\beta\alpha_2\left\{\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}\right\}}{\left[\left(\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}+k\right)\right][(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}$$

$$= -k\beta\alpha_2 \frac{\alpha_1-1}{(\gamma+\beta\alpha_1)\left(k-k(\alpha_1-1)\frac{\theta-\phi+1}{\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1}\right)(\theta-\phi+1)-k\beta\alpha_2(\alpha_1-1)\frac{\theta-\phi+1}{\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1}}$$

$$\frac{\theta-\phi+1}{\alpha_1+\alpha_2+(\alpha_1-1)(\theta-\phi)-1}$$

$$= -\beta\frac{\alpha_1-1}{\beta+\gamma}$$

$$n_3^*\big|_{h_t=\bar{h}_t} = \beta\frac{1-\alpha_1}{\beta+\gamma} \tag{C.27}$$

The following is observed through eq. C.19 - C.27:

(i) $n_1^*\big|_{h_t=\underline{h}_t} = n_2^*\big|_{h_t=\underline{h}_t}$

$$\beta\frac{[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)} \lesseqgtr \beta\frac{[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)}$$

As parameters on both sides of the equation are equal, hence it is concluded that both are equal.

(ii) $n_1^*\big|_{h_t=\bar{h}_t} > n_2^*\big|_{h_t=\bar{h}_t}$ since C.28 holds true.

$$\beta(1-\alpha_1)\frac{\theta-\phi+1}{\beta(1-\alpha_1-\alpha_2)(1-\phi)+\theta(\gamma+\beta)} > \beta\frac{1-\alpha_1}{\beta+\gamma}$$

$$\frac{\theta-\phi+1}{\beta(1-\alpha_1-\alpha_2)(1-\phi)+\theta(\gamma+\beta)} > \frac{1}{\beta+\gamma}$$

$$(\theta-\phi+1)(\beta+\gamma) > \beta(1-\alpha_1-\alpha_2)(1-\phi)+\theta(\gamma+\beta)$$

$$\theta(\gamma+\beta)-\phi(\gamma+\beta)+(\gamma+\beta) > \beta(1-\alpha_1-\alpha_2)(1-\phi)+\theta(\gamma+\beta)$$

$$(1-\phi)(\gamma+\beta) > \beta(1-\alpha_1-\alpha_2)(1-\phi)$$

$$(\gamma+\beta) > \beta(1-\alpha_1-\alpha_2)$$

$$\gamma > -\beta\alpha_1-\beta\alpha_2$$

$$\gamma+\beta\alpha_1+\beta\alpha_2 > 0 \tag{C.28}$$

(iii) $n_1^*\big|_{h_t=\underline{h}_t} > n_3^*\big|_{h_t=\underline{h}_t}$ since C.29 holds true.

$$\beta \frac{[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)} = \frac{\theta\beta(1-\alpha_1)+\beta\alpha_2(1-\phi)}{\theta(\beta+\gamma)+(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2)}$$
$$\frac{1}{\theta(\beta+\gamma)} > \frac{1}{\theta(\beta+\gamma)+(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2)}$$
$$[\theta(1-\alpha_1)+\alpha_2(1-\phi)][\theta(\beta+\gamma)+(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2)] >$$
$$[\theta\beta(1-\alpha_1)+\beta\alpha_2(1-\phi)]\,\theta(\beta+\gamma)$$
$$\theta(\beta+\gamma)+(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2) > \theta(\beta+\gamma)$$

$$(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2) > 0 \tag{C.29}$$

(iv) $n_1^*|_{h_t=\bar{h}_t} > n_3^*|_{h_t=\bar{h}_t}$ since C.30 holds true.

$$\beta(1-\alpha_1)\frac{\theta-\phi+1}{\beta(1-\alpha_1-\alpha_2)(1-\phi)+\theta(\gamma+\beta)} > \beta\frac{(1-\alpha_1)}{\beta+\gamma}$$
$$(\theta-\phi+1)(\beta+\gamma) > \beta(1-\alpha_1-\alpha_2)(1-\phi)+\theta(\gamma+\beta)$$
$$(1-\phi)(\beta+\gamma) > \beta(1-\alpha_1-\alpha_2)(1-\phi)$$
$$(\beta+\gamma)-\beta+\beta\alpha_1+\beta\alpha_2 > 0$$

$$\gamma+\beta\alpha_1+\beta\alpha_2 > 0 \tag{C.30}$$

(iii) $n_2^*\big|_{h_t=\underline{h}_t} > n_3^*\big|_{h_t=\underline{h}_t}$ since C.31 holds true.

$$\beta \frac{[\theta(1-\alpha_1)+\alpha_2(1-\phi)]}{\theta(\beta+\gamma)} > \frac{\theta\beta(1-\alpha_1)+\beta\alpha_2(1-\phi)}{\theta(\beta+\gamma)+(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2)}$$
$$\frac{1}{\theta(\beta+\gamma)} > \frac{1}{\theta(\beta+\gamma)+(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2)}$$
$$\theta(\beta+\gamma)+(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2) > \theta(\beta+\gamma)$$

$$(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2) > 0 \tag{C.31}$$

(iv) $n_2^*|_{h_t=\bar{h}_t} = n_3^*|_{h_t=\bar{h}_t}$

$$n_2^*|_{h_t=\bar{h}_t} \lesseqgtr n_3^*|_{h_t=\bar{h}_t}$$
$$\beta\frac{1-\alpha_1}{\beta+\gamma} \lesseqgtr \beta\frac{1-\alpha_1}{\beta+\gamma}$$

As parameters on both sides of the equation are equal and cancel out, hence it is concluded that both are equal.

(v) $n_1^* \Big|_{h_t=h_0=\frac{k}{\phi}} > n_2^* \Big|_{h_t=h_0=\frac{k}{\phi}}$ conditional upon C.32.

$$\frac{\beta\alpha_2}{(1+\phi)\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi^2} > \frac{\beta(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\left[\phi^2-\theta(1+\phi)\right]}$$
$$\alpha_2(\gamma+\beta)\left[\phi^2-\theta(1+\phi)\right] > (1-\alpha_1-\alpha_2)\beta\alpha_2\phi^2+(1-\alpha_1-\alpha_2)(1+\phi)\theta(\gamma+\beta\alpha_1)$$
$$\alpha_2(\gamma+\beta)\left[\phi^2-\theta-\theta\phi\right] > (1-\alpha_1-\alpha_2)\beta\alpha_2\phi^2+(1-\alpha_1-\alpha_2)(1+\phi)\theta(\gamma+\beta\alpha_1)$$
$$\alpha_2\gamma\phi^2+(\alpha_1+\alpha_2)\beta\alpha_2\phi^2 > (1-\alpha_1-\alpha_2)(1+\phi)\theta(\gamma+\beta\alpha_1)+\alpha_2(\gamma+\beta)\theta(1+\phi)$$
$$\alpha_2\phi^2(\gamma+\beta\alpha_1+\beta\alpha_2) > \theta(1+\phi)(\gamma+\beta\alpha_1+\beta\alpha_2)(1-\alpha_1)$$
$$\frac{\alpha_2\phi^2}{(1+\phi)(1-\alpha_1)} > \theta$$

$$\theta < \frac{\alpha_2\phi^2}{(1+\phi)(1-\alpha_1)} \tag{C.32}$$

We assume that C.31 holds true hence $n_1^* \Big|_{h_t=h_0=\frac{k}{\phi}} > n_2^* \Big|_{h_t=h_0=\frac{k}{\phi}}$

(vi) $n_1^* \Big|_{h_t=h_0=\frac{k}{\phi}} > n_3^* \Big|_{h_t=h_0=\frac{k}{\phi}}$ since C.33 holds true.

$$\frac{\beta\alpha_2}{(1+\phi)\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi^2} > \frac{\beta\alpha_2}{(1+\phi)(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2}$$
$$(1+\phi)(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2 > (1+\phi)\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi^2$$
$$(\gamma+\beta\alpha_1)(1+\phi)(1-\phi)+\beta\alpha_2 > \beta\alpha_2\phi^2$$
$$(\gamma+\beta\alpha_1)(1+\phi)(1-\phi) > \beta\alpha_2\phi^2-\beta\alpha_2$$
$$(\gamma+\beta\alpha_1)(1+\phi)(1-\phi) > -\beta\alpha_2(1-\phi^2)$$
$$(\gamma+\beta\alpha_1)(1+\phi)(1-\phi) > -\beta\alpha_2\left[(1)^2-(\phi)^2\right]$$
$$(\gamma+\beta\alpha_1)(1+\phi)(1-\phi) > -\beta\alpha_2(1+\phi)(1-\phi)$$

$$(\gamma+\beta\alpha_1+\beta\alpha_2) > 0 \tag{C.33}$$

Now it remains to be seen that in the last regime i.e. when $h_t > \bar{h}_t$, which of the three optimal solutions obtained for $n^*$ is $\lesseqgtr$ than each other. In order to observe this, we will look at the values of $h_t \longrightarrow \infty$ :

Determining $n_1^*|_{h_t \longrightarrow \infty}$

$$n_1^* = \frac{\beta\alpha_2 h_t}{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$

Dividing the numerator and denominator by $h_t$ :

$$= \frac{\beta\alpha_2 \frac{h_t}{h_t}}{\left(\frac{h_t}{h_t}+\frac{k}{h_t}\right)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\frac{\beta\alpha_2 k}{h_t}}$$

$$= \frac{\beta\alpha_2}{\left(1+\frac{k}{h_t}\right)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\frac{\beta\alpha_2 k}{h_t}}$$

Substituting $h_t \longrightarrow \infty$

$$n_1^*|_{h_t \longrightarrow \infty} \frac{\beta\alpha_2}{[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]} \tag{C.34}$$

Determining $n_2^*|_{h_t \longrightarrow \infty}$

$$n_2^* = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}$$

Dividing the numerator and denominator by $h_t$ :

$$= \frac{\beta \frac{h_t}{h_t}(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\left[\left(\frac{h_t}{h_t}+\frac{k}{h_t}\right)(\phi-\theta)-\frac{k}{h_t}\right]}$$

$$= \frac{\beta(1-\alpha_1-\alpha_2)}{(\gamma+\beta)\left[\left(1+\frac{k}{h_t}\right)(\phi-\theta)-\frac{k}{h_t}\right]}$$

Substituting $h_t \longrightarrow \infty$

$$n_2^*|_{h_t \longrightarrow \infty} = \frac{\beta(1-\alpha_1-\alpha_2)}{(\gamma+\beta)(\phi-\theta)} \tag{C.35}$$

Determining $n_3^*|_{h_t \longrightarrow \infty}$

$$n_3^* = \frac{\beta\alpha_2 h_t}{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}$$

Dividing the numerator and denominator by $h_t$ :

$$= \frac{\beta\alpha_2 \frac{h_t}{h_t}}{\left(\frac{h_t}{h_t}+\frac{k}{h_t}\right)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 \frac{k}{h_t}}$$

$$= \frac{\beta\alpha_2}{\left(1+\frac{k}{h_t}\right)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 \frac{k}{h_t}}$$

Substituting $h_t \longrightarrow \infty$

$$n_3^*|_{h_t \longrightarrow \infty} = \frac{\beta\alpha_2}{(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2} \tag{C.36}$$

The following is observed through eq. C.34-C.36:

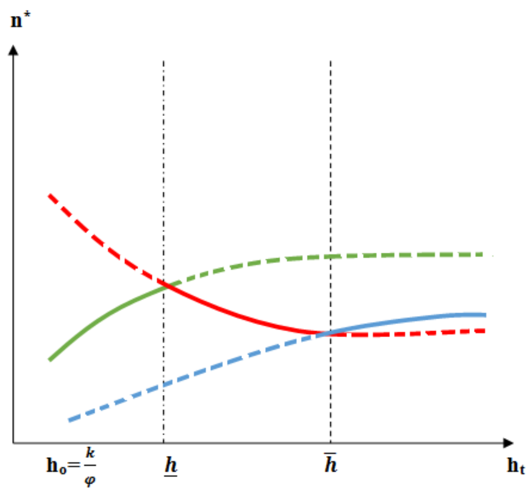
(vii) $n_1^*|_{h_t \longrightarrow \infty} > n_3^*|_{h_t \longrightarrow \infty}$ since C.37 holds true.

$$\frac{\beta\alpha_2}{[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]} > \frac{\beta\alpha_2}{(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2}$$
$$(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2 > \theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi$$
$$(\gamma+\beta\alpha_1)(1-\phi)+\beta\alpha_2 > \beta\alpha_2\phi$$
$$(\gamma+\beta\alpha_1)(1-\phi)+\beta\alpha_2-\beta\alpha_2\phi > 0$$
$$(\gamma+\beta\alpha_1)(1-\phi)+\beta\alpha_2(1-\phi) > 0$$

$$(1-\phi)(\gamma+\beta\alpha_1+\beta\alpha_2) > 0 \tag{C.37}$$

The analysis yields the following graphical representation of $n^*$ in all three regimes:

Combining the behavior of all three variables gives the following graphical representation:



# 10 Appendix D

This Appendix gives an illustration of the working of human capital accumulation and their graphical representation in all three regimes.

Given the human capital accumulation function as follows:

$$H_{t+1} = \left(\frac{c_t}{n_{t+1}}\right)^{\alpha_1} (e_{t+1} + \theta)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2} \tag{D.1}$$

In above plug in the optimal values of $e^*$and $N^* = \frac{c_t}{n_{t+1}}$ , hence the human capital accumulation function then is:

$$H_{t+1} = (N^*)^{\alpha_1} (e^* + \theta)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2} \tag{D.2}$$

(i) In order to calculate the human capital accumulation function in the first regime, plug in $e^* = 0$ and eq. A.11

$$H_{t+1} = (N^*)^{\alpha_1} (e^* + \theta)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$
$$H_{t+1} = \left(\frac{(h_t+k)\theta(\gamma+\beta\alpha_1)}{\beta\alpha_2}\right)^{\alpha_1} (0 + \theta)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$
$$H_{t+1} = \left(\frac{\theta(h_t+k)(\gamma+\beta\alpha_1)}{\beta\alpha_2}\right)^{\alpha_1} (\theta)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$

$$H_{t+1} = \frac{\theta^{\alpha_1+\alpha_2} (\gamma + \beta\alpha_1)^{\alpha_1} (h_t + k)^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}}{(\beta\alpha_2)^{\alpha_1}} \tag{D.3}$$

Taking first order derivative of equation D.3 where we suppose that $A = \frac{\theta^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{(\beta\alpha_2)^{\alpha_1}}$ hence, the equation then becomes:

$$H_{t+1} = A\,(h_t + k)^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}$$

$$H_{t+1} = A\left[h_t^{1-\alpha_2} + k^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}\right] \tag{D.4}$$

Taking the first order derivative of the above simplified equation:

$$\frac{\partial H_{t+1}}{\partial h_t} = A\left[(1-\alpha_2)\, h_t^{-\alpha_2} + (1-\alpha_1-\alpha_2)\, k^{\alpha_1} h_t^{-\alpha_1-\alpha_2}\right]$$
$$\frac{\partial H_{t+1}}{\partial h_t} = A\left[\frac{1-\alpha_2}{h_t^{\alpha_2}} + \frac{(1-\alpha_1-\alpha_2)k^{\alpha_1}}{h_t^{\alpha_1+\alpha_2}}\right]$$
$$\frac{\partial H_{t+1}}{\partial h_t} = A\left[\frac{(1-\alpha_2)h_t^{\alpha_1+\alpha_2}+(1-\alpha_1-\alpha_2)k^{\alpha_1}h_t^{\alpha_2}}{h_t^{\alpha_2}h_t^{\alpha_1+\alpha_2}}\right]$$
$$\frac{\partial H_{t+1}}{\partial h_t} = A\left[\frac{(1-\alpha_2)h_t^{\alpha_1+\alpha_2}+(1-\alpha_1-\alpha_2)k^{\alpha_1}h_t^{\alpha_2}}{h_t^{\alpha_1+2\alpha_2}}\right]$$
Plugging in $A = \frac{\theta^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{(\beta\alpha_2)^{\alpha_1}}$

$$\frac{\partial H_{t+1}}{\partial h_t} = \left(\frac{\theta^{\alpha_1+\alpha_2}\left(\gamma+\beta\alpha_1\right)^{\alpha_1}}{\left(\beta\alpha_2\right)^{\alpha_1}}\right)\left[\frac{(1-\alpha_2)\, h_t^{\alpha_1+\alpha_2} + (1-\alpha_1-\alpha_2)\, k^{\alpha_1} h_t^{\alpha_2}}{h_t^{\alpha_1+2\alpha_2}}\right] > 0 \quad \text{(D.5)}$$

Since $(1-\alpha_2)$, $(1-\alpha_1-\alpha_2)$, $(\gamma+\beta\alpha_1)$ and $\beta\alpha_2$ are $> 0$, and all other parameters $\theta$, $k$, and $h_t$ are positive, hence the above derivative is $> 0$.

Taking the second order derivative:

$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = A\left[(1-\alpha_2)\,(-\alpha_2)\, h_t^{-\alpha_2-1} + (1-\alpha_1-\alpha_2)\, k^{\alpha_1}\,(-\alpha_1-\alpha_2)\, h_t^{-\alpha_1-\alpha_2-1}\right]$$
$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = A\left[-\frac{(1-\alpha_2)\alpha_2}{h_t^{1+\alpha_2}} - \frac{(1-\alpha_1-\alpha_2)k^{\alpha_1}(\alpha_1+\alpha_2)}{h1_t^{1+\alpha_1+\alpha_2}}\right]$$
$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = -A\left[\frac{(1-\alpha_2)\alpha_2}{h_t^{1+\alpha_2}} + \frac{(1-\alpha_1-\alpha_2)k^{\alpha_1}(\alpha_1+\alpha_2)}{h1_t^{1+\alpha_1+\alpha_2}}\right]$$
Plugging in $A = \frac{\theta^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{(\beta\alpha_2)^{\alpha_1}}$

$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = -\left(\frac{\theta^{\alpha_1+\alpha_2}\left(\gamma+\beta\alpha_1\right)^{\alpha_1}}{\left(\beta\alpha_2\right)^{\alpha_1}}\right)\left[\frac{(1-\alpha_2)\,\alpha_2}{h_t^{1+\alpha_2}} + \frac{(1-\alpha_1-\alpha_2)\, k^{\alpha_1}\,(\alpha_1+\alpha_2)}{h_t^{1+\alpha_1+\alpha_2}}\right] < 0 \quad \text{(D.6)}$$

Since $(1-\alpha_2)$, $(1-\alpha_1-\alpha_2)$, $(\gamma+\beta\alpha_1)$ and $\beta\alpha_2$ are $> 0$, and all other parameters $\theta$, $k$, and $h_t$ are positive, hence due to the negative sign with the above derivative, $\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} < 0$

(ii) In order to calculate the human capital accumulation function in the second regime, plug in A.4 and A.13

$$H_{t+1} = (N^*)^{\alpha_1}\,(e^*+\theta)^{\alpha_2}\, h_t^{1-\alpha_1-\alpha_2}$$
$$H_{t+1} = \left(\frac{(\gamma+\beta\alpha_1)[(h_t+k)(\phi-\theta)-k]}{\beta(1-\alpha_1-\alpha_2)}\right)^{\alpha_1}\left(\frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}+\theta\right)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$
$$H_{t+1} =$$
$$\left(\frac{(\gamma+\beta\alpha_1)[\phi h_t-k(1-\phi)-\theta(h_t+k)]}{\beta(1-\alpha_1-\alpha_2)}\right)^{\alpha_1}\left(\frac{[\phi h_t-k(1-\phi)]-\theta(1-\alpha_1)(h_t+k)+\theta(1-\alpha_1)(h_t+k)-\theta\alpha_2(h_t+k)}{(h_t+k)(1-\alpha_1-\alpha_2)}\right)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$

$$H_{t+1} = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}[\phi h_t - k(1-\phi)-\theta(h_t+k)]^{\alpha_1}(\alpha_2[\phi h_t - k(1-\phi)-\theta(h_t+k)]^{\alpha_2})h_t^{1-\alpha_1-\alpha_2}}{\beta^{\alpha_1}(h_t+k)^{\alpha_2}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}$$

$$H_{t+1} = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\left[\phi h_t - k(1-\phi) - \theta(h_t+k)\right]^{\alpha_1+\alpha_2} h_t^{1-\alpha_1-\alpha_2}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}(h_t+k)^{\alpha_2}} \tag{D.7}$$

Suppose that in equation D.7, $V = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}$ hence, the equation then becomes:

$$H_{t+1} = V\frac{[\phi h_t - k(1-\phi)-\theta(h_t+k)]^{\alpha_1+\alpha_2} h_t^{1-\alpha_1-\alpha_2}}{(h_t+k)^{\alpha_2}}$$

Simplifying the above equation:

$$H_{t+1} = V\frac{[h_t(\phi-\theta)-k(1-\phi+\theta)]^{\alpha_1+\alpha_2} h_t^{1-\alpha_1-\alpha_2}}{(h_t+k)^{\alpha_2}}$$

Taking logarithms:

$$\log H_{t+1} =$$

$$\log V + (\alpha_1+\alpha_2)\log\left[h_t(\phi-\theta) - k(1-\phi+\theta)\right] + (1-\alpha_1-\alpha_2)h_t - \alpha_2\log(h_t+k)$$

$$\log H_{t+1} = \log V + (1-\alpha_2)\log h_t + (\alpha_1+\alpha_2)\log(\phi-\theta) - (\alpha_1+\alpha_2)\log k -$$

$$(\alpha_1+\alpha_2)\log(1-\phi+\theta) - \alpha_2\log k$$

Taking exponentials:

$$H_{t+1} = V\frac{(\phi-\theta)^{(\alpha_1+\alpha_2)}}{k^{(\alpha_1+2\alpha_2)}(1-\phi+\theta)^{(\alpha_1+\alpha_2)}}$$

where we suppose that $X = \left[\frac{(\phi-\theta)}{k(1-\phi+\theta)}\right]^{\alpha_1+\alpha_2}$

Hence, the equation then becomes:

$$H_{t+1} = VXh_t^{(1-\alpha_2)}$$

Taking the first order derivative of the above equation:

$$\frac{\partial H_{t+1}}{\partial h_t} = \frac{VX(1-\alpha_2)}{h_t^{\alpha_2}}$$

Substituting the value of $V$ and $X$ :

$$\frac{\partial H_{t+1}}{\partial h_t} = \frac{\alpha_2^{\alpha_2}\left(\gamma+\beta\alpha_1\right)^{\alpha_1}}{\beta^{\alpha_1}\left(1-\alpha_1-\alpha_2\right)^{\alpha_1+\alpha_2}}\left[\frac{(\phi-\theta)}{k\left(1-\phi+\theta\right)}\right]^{\alpha_1+\alpha_2}\frac{\left(1-\alpha_2\right)}{h_t^{\alpha_2}} > 0 \tag{D.8}$$

Taking the second order derivative:

$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = -\frac{VX(1-\alpha_2)(\alpha_2)}{h_t^{1+\alpha_2}}$$

Substituting the value of $V$ and $X$ :

$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = -\frac{\left(1-\alpha_2\right)\left(\alpha_2\right)}{h_t^{1+\alpha_2}}\frac{\alpha_2^{\alpha_2}\left(\gamma+\beta\alpha_1\right)^{\alpha_1}}{\beta^{\alpha_1}\left(1-\alpha_1-\alpha_2\right)^{\alpha_1+\alpha_2}}\left[\frac{(\phi-\theta)}{k\left(1-\phi+\theta\right)}\right]^{\alpha_1+\alpha_2} < 0 \tag{D.9}$$

Since it has been previously been established that $(\phi-\theta) > 0$ and $(1-\phi+\theta) > 0$ and all other parameters are positive hence in the second regime $\frac{\partial H_{t+1}}{\partial h_t} > 0$ and $\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} < 0$.

(iii) In order to calculate the human capital accumulation function in the third regime, plug in eq. A.14 and $e^* = 1-\phi$ :

$$H_{t+1} = \left(N^*\right)^{\alpha_1}\left(e^*+\theta\right)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$
$$H_{t+1} = \left(\frac{(h_t+k)(1-\phi+\theta)(\gamma+\beta\alpha_1)}{\beta\alpha_2}\right)^{\alpha_1}\left(1-\phi+\theta\right)^{\alpha_2} h_t^{1-\alpha_1-\alpha_2}$$

$$H_{t+1} = \frac{\left(\gamma+\beta\alpha_1\right)^{\alpha_1}\left(1+\theta-\phi\right)^{\alpha_1+\alpha_2}\left(h_t+k\right)^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}}{\left(\beta\alpha_2\right)^{\alpha_1}} \tag{D.10}$$

Suppose that in equation D.10, $Z = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$, hence the equation then becomes:

$$H_{t+1} = Z\left[\left(h_t+k\right)^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}\right]$$
$$H_{t+1} = Z\left[h_t^{1-\alpha_2} + k^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}\right]$$
$$\frac{\partial H_{t+1}}{\partial h_t} = Z\left[\left(1-\alpha_2\right) h_t^{-\alpha_2} + k^{\alpha_1}\left(1-\alpha_1-\alpha_2\right) h_t^{-\alpha_1-\alpha_2}\right]$$
$$\frac{\partial H_{t+1}}{\partial h_t} = Z\left[\frac{(1-\alpha_2)}{h_t^{\alpha_2}} + \frac{k^{\alpha_1}(1-\alpha_1-\alpha_2)}{h_t^{\alpha_1+\alpha_2}}\right]$$

Plugging in $Z = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$

$$\frac{\partial H_{t+1}}{\partial h_t} = \left( \frac{(\gamma + \beta\alpha_1)^{\alpha_1} (1 + \theta - \phi)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1}} \right) \left[ \frac{(1 - \alpha_2)}{h_t^{\alpha_2}} + \frac{k^{\alpha_1} (1 - \alpha_1 - \alpha_2)}{h_t^{\alpha_1+\alpha_2}} \right] > 0 \quad \text{(D.11)}$$

Since all parameters are $> 0$, hence D.11 is $> 0$.

Taking the second order derivative:

$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = Z \left[ -\alpha_2 (1 - \alpha_2) h_t^{-\alpha_2-1} + k^{\alpha_1} (1 - \alpha_1 - \alpha_2) (-\alpha_1 - \alpha_2) h_t^{-\alpha_1-\alpha_2-1} \right]$$

$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = -Z \left[ \frac{\alpha_2(1-\alpha_2)}{h_t^{1+\alpha_2}} + \frac{k^{\alpha_1}(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2)}{h_t^{1+\alpha_1+\alpha_2}} \right]$$

Plugging in $Z = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$

$$\frac{\partial H_{t+1}/\partial h_t}{\partial h_t} = -\left( \frac{(\gamma + \beta\alpha_1)^{\alpha_1} (1 + \theta - \phi)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1}} \right) \left[ \frac{\alpha_2 (1 - \alpha_2)}{h_t^{1+\alpha_2}} + \frac{k^{\alpha_1} (1 - \alpha_1 - \alpha_2) (\alpha_1 + \alpha_2)}{h_t^{1+\alpha_1+\alpha_2}} \right] < 0 \quad \text{(D.12)}$$

Since all parameters are positive, hence due to the negative sign with the derivative, D.12 $< 0$.

# 11 Appendix E

This Appendix gives an illustration of the working of steady-states in all three regimes and provides proof of the steady-states remaining in the limits given:

To derive the steady state equilibrium in the three regimes equate $h_t = H_{t+1}$, substitue the human capital accumulation function of the respective regime and then solve for $h_t$.

**Steady state of the first regime** $h_{s1}$ :

Substitute $H_{t+1} = \frac{\theta^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}(h_t+k)^{\alpha_1}h_t^{1-\alpha_1-\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$ as given in eq. D.3 in the following equation:

$$h_t = H_{t+1}$$

$$h_t = \frac{\theta^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}(h_t+k)^{\alpha_1}h_t^{1-\alpha_1-\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$$

Take logs on both sides of the equation:

$$\log h_t = \left[\log\theta^{\alpha_1+\alpha_2} + \log(\gamma+\beta\alpha_1)^{\alpha_1} + \log(h_t+k)^{\alpha_1} + \log h_t^{1-\alpha_1-\alpha_2}\right] - \log(\beta\alpha_2)^{\alpha_1}$$

$$\log h_t = [(\alpha_1+\alpha_2)\log\theta + \alpha_1\log(\gamma+\beta\alpha_1) + \alpha_1\log h_t + \alpha_1\log k + (1-\alpha_1-\alpha_2)\log h_t] - \alpha_1\log(\beta\alpha_2)$$

$$\log h_t - \alpha_1\log h_t - (1-\alpha_1-\alpha_2)\log h_t =$$

$$(\alpha_1+\alpha_2)\log\theta + \alpha_1\log(\gamma+\beta\alpha_1) + \alpha_1\log k - \alpha_1\log(\beta\alpha_2)$$

$$\log\left(\frac{h_t}{h_t^{\alpha_1}h_t^{1-\alpha_1-\alpha_2}}\right) = \log\theta^{\alpha_1+\alpha_2} + \log(\gamma+\beta\alpha_1)^{\alpha_1} + \log k^{\alpha_1} - \log(\beta\alpha_2)^{\alpha_1}$$

Taking exponential

$$h_t^{\alpha_2} = \frac{\theta^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}k^{\alpha_1}}{(\beta\alpha_2)^{\alpha_1}}$$

Dividing the power by $\alpha_2$ on both sides to get $h_{s1}$

$$h_{s1} = \left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}\theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}} \tag{E.1}$$

**Steady state of the second regime** $h_{s2}$ :

Substitute $H_{t+1} = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}[\phi h_t - k(1-\phi)-\theta(h_t+k)]^{\alpha_1+\alpha_2}h_t^{1-\alpha_1-\alpha_2}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}(h_t+k)^{\alpha_2}}$ as given in eq. D.7 in $h_t = H_{t+1}$ :

$h_t = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}[\phi h_t - k(1-\phi)-\theta(h_t+k)]^{\alpha_1+\alpha_2}h_t^{1-\alpha_1-\alpha_2}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}(h_t+k)^{\alpha_2}}$ where assume $Y = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}$

Hence the equation becomes $h_t = \frac{Y[\phi h_t - k(1-\phi) - \theta(h_t+k)]^{\alpha_1+\alpha_2} h_t^{1-\alpha_1-\alpha_2}}{(h_t+k)^{\alpha_2}}$

Taking logs:

$$\log h_t = \log Y + (1-\alpha_1-\alpha_2)\log$$

$$h_t + (\alpha_1+\alpha_2)\log\left[\phi h_t - k(1-\phi) - \theta(h_t+k)\right] - \alpha_2 \log(h_t+k)$$

$$\log h_t = \log Y + (1-\alpha_1-\alpha_2)\log$$

$$h_t + (\alpha_1+\alpha_2)\log \phi h_t - (\alpha_1+\alpha_2)\log k(1-\phi) - (\alpha_1+\alpha_2)\log\theta(h_t+k)$$

$$-\alpha_2 \log(h_t+k)$$

$$\log h_t - (1-\alpha_1-\alpha_2)\log\ h_t - (\alpha_1+\alpha_2)\log h_t + (\alpha_1+\alpha_2)\log h_t + \alpha_2 \log h_t$$

$$=$$

$$\log Y + (\alpha_1+\alpha_2)\log\left[\phi - k(1-\phi) - \theta - \theta k\right] - \alpha_2 \log k$$

$$\log\left(\frac{h_t h_t^{(\alpha_1+\alpha_2)} h_t^{\alpha_2}}{h_t^{1-\alpha_1-\alpha_2} h_t^{(\alpha_1+\alpha_2)}}\right) = \log Y + \log\left[\phi - k(1-\phi) - \theta(1+k)\right]^{\alpha_1+\alpha_2} - \log k^{\alpha_2}$$

$$\log h_t^{2\alpha_2+\alpha_1} = \log Y + \log\left[\phi - k(1-\phi) - \theta(1+k)\right]^{\alpha_1+\alpha_2} - \log k^{\alpha_2}$$

Taking exponential,

$$h_t^{2\alpha_2+\alpha_1} = \frac{[\phi - k(1-\phi) - \theta(1+k)]^{\alpha_1+\alpha_2}}{Y k^{\alpha_2}}$$

Dividing the power by $(2\alpha_2+\alpha_1)$ on both sides and substituting $Y = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}$

$$h_{s2} = \left[\frac{[\phi - k(1-\phi) - \theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}} \left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1} k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}} \tag{E.2}$$

**Steady state of the third regime** $h_{s3}$ :

Substitute $H_{t+1} = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}(h_t+k)^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$ as given in eq. D.10 in $h_t = H_{t+1}$ :

$$h_t = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}(h_t+k)^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}}{(\beta\alpha_2)^{\alpha_1}} \text{where assume } A = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$$

Hence the equation becomes $h_t = A(h_t+k)^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}$

Taking logs:

$$\log h_t = \log A + \alpha_1 \log(h_t+k) + (1-\alpha_1-\alpha_2)\log h_t$$

$$\log h_t = \log A + \alpha_1 \log h_t + \alpha_1 \log k + (1-\alpha_1-\alpha_2)\log h_t$$

$$\log h_t - \alpha_1 \log h_t - (1-\alpha_1-\alpha_2)\log h_t = \log A + \alpha_1 \log k$$

$$\log\left(\frac{h_t}{h_t^{\alpha_1} h_t^{1-\alpha_1-\alpha_2}}\right) = \log A + \alpha_1 \log k$$

$$\log h_t^{\alpha_2} = \log A + \log k^{\alpha_1}$$

Taking exponential

$$h_t^{\alpha_2} = Ak^{\alpha_1}$$

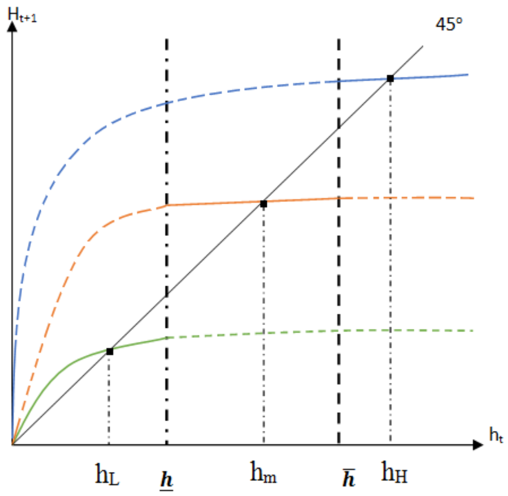
Dividing the power by $\alpha_2$ on both sides

$$h_t = A^{\frac{1}{\alpha_2}} k^{\frac{\alpha_1}{\alpha_2}}$$

Substitute $A = \frac{(\gamma+\beta\alpha_1)^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1}}$

$$h_{s3} = \left[\frac{(\gamma+\beta\alpha_1)\,k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}} (1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}} \tag{E.3}$$

The working of the derivatives of $H_{t+1}$ with respect to $h_t$ given in Appendix D yields the following graph of the steady-state:




In this paper we take:

(i) $h_{s1}$ as the low steady-state and denote it as $h_L$.

(ii) $h_{s2}$ as the medium steady-state and denote it as $h_m$.

(iii) $h_{s3}$ as the high steady-state and denote it as $h_H$.

Now we need to ensure that the steady-states remain in the following regimes:

$$h_L < \underline{h}_t$$

$$\underline{h}_t < h_m < \bar{h}_t$$

$$h_H > \bar{h}_t$$

Ensuring that $h_L < \underline{h}_t$ :

$$\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}} \theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}} < \frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}$$

Multiplying the power by $\alpha_2$ on both sides:

$$\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\alpha_1} \theta^{\alpha_1+\alpha_2} < \left\{\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}\right\}^{\alpha_2}$$

Taking logs on both sides:

$$\alpha_1 \log\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right] + (\alpha_1+\alpha_2)\,\theta < \alpha_2 \log\left\{\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}\right\}$$

$$\alpha_1 \log(\gamma+\beta\alpha_1) + \alpha_1 \log k - \alpha_1 \log\beta\alpha_2 + (\alpha_1+\alpha_2)\log\theta <$$

$$\alpha_2 \log k+\alpha_2 \log\alpha_2+\alpha_2 \log(1-\phi)+\alpha_2 \log\theta+\alpha_2 \log(1-\alpha_1)-\alpha_2 \log\alpha_2\phi+\alpha_2 \log\theta\,(1-\alpha_1)$$

$$\alpha_1 \log(\gamma+\beta\alpha_1) + \alpha_1 \log k - \alpha_1 \log\beta\alpha_2 + (\alpha_1+\alpha_2)\log\theta$$

$$<$$

$$\alpha_2 \log k + \alpha_2 \log\alpha_2 + \alpha_2 \log 1 - \alpha_2 \log\phi + \alpha_2 \log\theta + \alpha_2 \log(1-\alpha_1) - \alpha_2 \log\alpha_2 -$$

$$\alpha_2 \log\phi + \alpha_2 \log\theta + \alpha_2 \log(1-\alpha_1)$$

$$(\alpha_1-\alpha_2)\log\theta + 2\alpha_2 \log\phi <$$

$$(\alpha_2-\alpha_1)\log k + 2\alpha_2 \log(1-\alpha_1) - \alpha_1 \log(\gamma+\beta\alpha_1) + \alpha_1 \log\beta\alpha_2$$

$$\log\theta^{\alpha_1-\alpha_2} + \log\phi^{2\alpha_2} < \log\left[\frac{k^{\alpha_2-\alpha_1}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}}\right]$$

Taking exponentials:

$$\theta^{\alpha_1-\alpha_2}\phi^{2\alpha_2} < \frac{k^{\alpha_2-\alpha_1}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}}$$

$$\theta^{\alpha_1-\alpha_2} < \frac{k^{\alpha_2-\alpha_1}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{2\alpha_2}}$$

Dividing the power by $(\alpha_1 - \alpha_2)$ on both sides:

$$\theta < \left[\frac{k^{\alpha_2-\alpha_1}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{2\alpha_2}}\right]^{\frac{1}{\alpha_1-\alpha_2}} \tag{E.4}$$

Ensuring that $\underline{h}_t < h_m$ :

$$\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)} < \left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}} \left[\frac{\alpha_2^{\alpha_2 2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}$$

In the above inequality we suppose that $A = \frac{\alpha_2^{\alpha_2 2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}$

Hence, the equation then becomes:

$$\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)} < \left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}} [A]^{\frac{1}{2\alpha_2+\alpha_1}}$$

Multiplying the power by $(2\alpha_2 + \alpha_1)$ :

$$\left\{\frac{k[\alpha_2(1-\phi)+\theta(1-\alpha_1)]}{\alpha_2\phi-\theta(1-\alpha_1)}\right\}^{(2\alpha_2+\alpha_1)} < A\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\alpha_1+\alpha_2}$$

Taking logs on both sides of the inequality:

$$(2\alpha_2+\alpha_1)\log k + (2\alpha_2+\alpha_1)\log\alpha_2 + (2\alpha_2+\alpha_1)\log 1 - (2\alpha_2+\alpha_1)\log\phi + (2\alpha_2+\alpha_1)\log\theta$$
$$+ (2\alpha_2+\alpha_1)\log(1-\alpha_1) - (2\alpha_2+\alpha_1)\log\alpha_2 - (2\alpha_2+\alpha_1)\log\phi + (2\alpha_2+\alpha_1)\log\theta +$$
$$(2\alpha_2+\alpha_1)\log(1-\alpha_1)$$
$$<$$
$$\log A + (\alpha_1+\alpha_2)\log\phi - (\alpha_1+\alpha_2)\log k + (\alpha_1+\alpha_2)\log k + (\alpha_1+\alpha_2)\log\phi -$$
$$(\alpha_1+\alpha_2)\log\theta - (\alpha_1+\alpha_2)\log\theta$$
$$-(\alpha_1+\alpha_2)\log k - (\alpha_1+\alpha_2)\log(1-\alpha_1-\alpha_2)$$

$$(2\alpha_2+\alpha_1)\log k - 2(2\alpha_2+\alpha_1)\log\phi + 2(2\alpha_2+\alpha_1)\log\theta + 2(2\alpha_2+\alpha_1)\log(1-\alpha_1)$$
$$<$$
$$\log A - 2(\alpha_1+\alpha_2)\log\phi - (\alpha_1+\alpha_2)\log k - 2(\alpha_1+\alpha_2)\log\theta - (\alpha_1+\alpha_2)\log(1-\alpha_1-\alpha_2)$$

$$(6\alpha_2+4\alpha_1)\log\theta - 2\alpha_2\log\phi$$
$$<$$
$$\log A - (2\alpha_1+3\alpha_2)\log k - (\alpha_1+\alpha_2)\log(1-\alpha_1-\alpha_2) - (4\alpha_2+2\alpha_1)\log(1-\alpha_1)$$

Taking exponentials:

$$\frac{\theta^{(6\alpha_2+4\alpha_1)}}{\phi^{(2\alpha_2)}} < \frac{A}{k^{(2\alpha_1+3\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}$$

Substituting $A = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}$

$$\frac{\theta^{(6\alpha_2+4\alpha_1)}}{\phi^{(2\alpha_2)}} < \frac{\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}}{k^{(2\alpha_1+3\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}$$
$$\frac{\theta^{(6\alpha_2+4\alpha_1)}}{\phi^{(2\alpha_2)}} < \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{(2\alpha_1+4\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}$$
$$\theta^{(6\alpha_2+4\alpha_1)} < \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(2\alpha_2)}}{\beta^{\alpha_1}k^{(2\alpha_1+4\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}$$

Dividing the power by $(6\alpha_2 + 4\alpha_1)$ :

$$\theta < \left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(2\alpha_2)}}{\beta^{\alpha_1}k^{(2\alpha_1+4\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}\right]^{\frac{1}{(6\alpha_2+4\alpha_1)}}$$

$$\theta < \left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(2\alpha_2)}}{\beta^{\alpha_1}k^{(2\alpha_1+4\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}\right]^{\frac{1}{(6\alpha_2+4\alpha_1)}} \tag{E.5}$$

Ensuring $h_m < \bar{h}_t$ :

$$\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}} \left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}} < \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

where we suppose $A = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}$

Hence, the equation then becomes:

$$\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}} [A]^{\frac{1}{2\alpha_2+\alpha_1}} < \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

Multiplying the power by $(2\alpha_2 + \alpha_1)$ on both sides:

$$A\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\alpha_1+\alpha_2} < \left\{\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}\right\}^{2\alpha_2+\alpha_1}$$

Taking logs:

$$\log A + (\alpha_1+\alpha_2)\log\phi - (\alpha_1+\alpha_2)\log k - (\alpha_1+\alpha_2)\log(1-\phi)$$
$$-(\alpha_1+\alpha_2)\log\theta - (\alpha_1+\alpha_2)\log(1+k) - (\alpha_1+\alpha_2)\log(1-\alpha_1-\alpha_2)$$
$$<$$
$$(2\alpha_2+\alpha_1)\log k + (2\alpha_2+\alpha_1)\log(1-\alpha_1) + (2\alpha_2+\alpha_1)\log(1+\theta-\phi)$$
$$-(2\alpha_2+\alpha_1)\log\alpha_2 + (2\alpha_2+\alpha_1)\log(1-\alpha_1) + (2\alpha_2+\alpha_1)\log(1+\theta-\phi)$$
$$\log A + 2(\alpha_1+\alpha_2)\log\phi - 2(\alpha_1+\alpha_2)\log k - (\alpha_1+\alpha_2)\log\theta - (\alpha_1+\alpha_2)\log(1-\alpha_1-\alpha_2)$$

$$<$$

$$(2\alpha_2 + \alpha_1)\log k + 2(2\alpha_2 + \alpha_1)\log(1-\alpha_1) + 2(2\alpha_2 + \alpha_1)\log(1+\theta-\phi) -$$

$$(2\alpha_2 + \alpha_1)\log\alpha_2$$

$$2(\alpha_1 + \alpha_2)\log\phi - (\alpha_1 + \alpha_2)\log\theta - 2(2\alpha_2 + \alpha_1)\log\theta + 2(2\alpha_2 + \alpha_1)\log\phi$$

$$<$$

$$(2\alpha_2 + \alpha_1)\log k + 2(2\alpha_2 + \alpha_1)\log(1-\alpha_1) - (2\alpha_2 + \alpha_1)\log\alpha_2 - \log A +$$

$$2(\alpha_1 + \alpha_2)\log k + (\alpha_1 + \alpha_2)\log(1-\alpha_1-\alpha_2)$$

$$(6\alpha_2 + 4\alpha_1)\log\phi - (5\alpha_2 + 3\alpha_1)\log\theta$$

$$<$$

$$(4\alpha_2 + 3\alpha_1)\log k + (4\alpha_2 + 2\alpha_1)\log(1-\alpha_1) - (2\alpha_2 + \alpha_1)\log\alpha_2 - \log A +$$

$$(\alpha_1 + \alpha_2)\log(1-\alpha_1-\alpha_2)$$

Taking exponentials:

$$\frac{\phi^{(6\alpha_2+4\alpha_1)}}{\theta^{(5\alpha_2+3\alpha_1)}} < \frac{k^{(4\alpha_2+3\alpha_1)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}}{\alpha_2^{(2\alpha_2+\alpha_1)}A}$$

$$\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}A < k^{(4\alpha_2+3\alpha_1)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}\theta^{(5\alpha_2+3\alpha_1)}$$

$$\frac{\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}A}{k^{(4\alpha_2+3\alpha_1)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}} < \theta^{(5\alpha_2+3\alpha_1)}$$

Substituting value of $A = \frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}$

$$\frac{\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}}{k^{(4\alpha_2+3\alpha_1)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}} < \theta^{(5\alpha_2+3\alpha_1)}$$

$$\frac{\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{(5\alpha_2+3\alpha_1)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}} < \theta^{(5\alpha_2+3\alpha_1)}$$

Dividing the power by $(5\alpha_2 + 3\alpha_1)$ on both sides:

$$\left[\frac{\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}}\right]^{\frac{1}{(5\alpha_2+3\alpha_1)}}\left(\frac{1}{k}\right) < \theta$$

$$\theta > \left[\frac{\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}}\right]^{\frac{1}{(5\alpha_2+3\alpha_1)}}\left(\frac{1}{k}\right) \tag{E.6}$$

Ensuring $h_H > \bar{h}_t$ :

$$\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}} > \frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}$$

Multiplying the power by $\alpha_2$ on both sides:

$$\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\alpha_1}(1+\theta-\phi)^{\alpha_1+\alpha_2} > \left\{\frac{k(1-\alpha_1)(1+\theta-\phi)}{[\alpha_2-(1-\alpha_1)(1+\theta-\phi)]}\right\}^{\alpha_2}$$

Taking logs on both sides:

$$\alpha_1 \log(\gamma+\beta\alpha_1)+\alpha_1\log k-\alpha_1\log(\beta\alpha_2)+(\alpha_1+\alpha_2)\log 1+(\alpha_1+\alpha_2)\log\theta-(\alpha_1+\alpha_2)\log\phi$$

$$>$$

$$\alpha_2\log k+\alpha_2\log(1-\alpha_1)+\alpha_2\log 1+\alpha_2\log\theta-\alpha_2\log\phi-\alpha_2\log\alpha_2+\alpha_2\log(1-\alpha_1)+$$

$$\alpha_2\log 1+\alpha_2\log\theta-\alpha_2\log\phi$$

$$(\alpha_1-\alpha_2)\log\theta+(\alpha_2-\alpha_1)\log\phi$$

$$>$$

$$(\alpha_2-\alpha_1)\log k+2\alpha_2\log(1-\alpha_1)-\alpha_2\log\alpha_2+\alpha_1\log(\beta\alpha_2)-\alpha_1\log(\gamma+\beta\alpha_1)$$

Taking exponentials:

$$\theta^{(\alpha_1-\alpha_2)}\phi^{(\alpha_2-\alpha_1)} > \frac{k^{(\alpha_2-\alpha_1)}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{\alpha_2^{\alpha_2}{}^2(\gamma+\beta\alpha_1)^{\alpha_1}}$$

$$\theta^{(\alpha_1-\alpha_2)} > \frac{k^{(\alpha_2-\alpha_1)}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{\alpha_2^{\alpha_2}{}^2(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(\alpha_2-\alpha_1)}}$$

Dividing the power by $(\alpha_1-\alpha_2)$ :

$$\theta > \left[\frac{k^{(\alpha_2-\alpha_1)}(1-\alpha_1)^{2\alpha_2}}{\alpha_2^{\alpha_2}{}^2\phi^{(\alpha_2-\alpha_1)}}\left(\frac{\beta\alpha_2}{\gamma+\beta\alpha_1}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1-\alpha_2}} \tag{E.7}$$

Thus, we have the following four inequalities:

(I) $\theta < \left[\frac{k^{\alpha_2-\alpha_1}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{2\alpha_2}}\right]^{\frac{1}{\alpha_1-\alpha_2}}$

(II) $\theta < \left[\frac{\alpha_2^{\alpha_2}{}^2(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(2\alpha_2)}}{\beta^{\alpha_1}k^{(2\alpha_1+4\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}\right]^{\frac{1}{(6\alpha_2+4\alpha_1)}}$

(III) $\theta > \left[\frac{\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}\alpha_2^{\alpha_2}{}^2(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}}\right]^{\frac{1}{(5\alpha_2+3\alpha_1)}}\left(\frac{1}{k}\right)$

(IV) $\theta > \left[\frac{k^{(\alpha_2-\alpha_1)}(1-\alpha_1)^{2\alpha_2}}{\alpha_2^{\alpha_2}{}^2\phi^{(\alpha_2-\alpha_1)}}\left(\frac{\beta\alpha_2}{\gamma+\beta\alpha_1}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1-\alpha_2}}$

Now we need to see if inequality (I) $\lesseqgtr$ Inequality (II):

$$\left[\frac{k^{\alpha_2-\alpha_1}(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{2\alpha_2}}\right]^{\frac{1}{\alpha_1-\alpha_2}} \lesseqgtr \left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(2\alpha_2)}}{\beta^{\alpha_1}k^{(2\alpha_1+4\alpha_2)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{4\alpha_2+2\alpha_1}}\right]^{\frac{1}{(6\alpha_2+4\alpha_1)}}$$

$$\left[\frac{(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{2\alpha_2}}\right]^{\frac{1}{\alpha_1-\alpha_2}} \lesseqgtr \left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(2\alpha_2)}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}}\right]^{\frac{1}{(6\alpha_2+4\alpha_1)}}\left(\frac{1}{k^{\frac{(2\alpha_1+4\alpha_2)}{(6\alpha_2+4\alpha_1)}}}\right)$$

$$\left[\frac{(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{2\alpha_2}}\right]^{\frac{1}{\alpha_1-\alpha_2}} \lesseqgtr \left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(2\alpha_2)}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}}\right]^{\frac{1}{(6\alpha_2+4\alpha_1)}} k^{\frac{(\alpha_1+\alpha_2)}{2\alpha_1+3\alpha_2}}$$

$$\left[\frac{(1-\alpha_1)^{2\alpha_2}(\beta\alpha_2)^{\alpha_1}}{(\gamma+\beta\alpha_1)^{\alpha_1}}\right]^{\frac{1}{\alpha_1-\alpha_2}} \lesseqgtr$$

$$\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}}\right]^{\frac{1}{(6\alpha_2+4\alpha_1)}} \phi^{\frac{(2\alpha_2)}{(6\alpha_2+4\alpha_1)}}\phi^{\frac{2\alpha_2}{\alpha_1-\alpha_2}} k^{\frac{(\alpha_1+\alpha_2)}{2\alpha_1+3\alpha_2}}$$

$$\left[\frac{(1-\alpha_1)^{\frac{2\alpha_2}{\alpha_1-\alpha_2}}(\beta\alpha_2)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}}{(\gamma+\beta\alpha_1)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}}\right] \lesseqgtr$$

$$\left[\frac{\alpha_2^{\frac{\alpha_2}{(6\alpha_2+4\alpha_1)}}(\gamma+\beta\alpha_1)^{\frac{\alpha_1}{(6\alpha_2+4\alpha_1)}}}{\beta^{\frac{\alpha_1}{(6\alpha_2+4\alpha_1)}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1+\alpha_2)}{(6\alpha_2+4\alpha_1)}}(1-\alpha_1)^{\frac{(4\alpha_2+2\alpha_1)}{(6\alpha_2+4\alpha_1)}}}\right]\phi^{5\alpha_2\frac{\alpha_1+\alpha_2}{2\alpha_1^2+\alpha_1\alpha_2-3\alpha_2^2}} k^{\frac{(\alpha_1+\alpha_2)}{2\alpha_1+3\alpha_2}}$$

$$\frac{\beta^{\frac{5}{2}\alpha_1\frac{\alpha_1+\alpha_2}{2\alpha_1^2+\alpha_1\alpha_2-3\alpha_2^2}}(1-\alpha_1)^{\frac{\alpha_1^2+5\alpha_1\alpha_2+4\alpha_2^2}{2\alpha_1^2+\alpha_1\alpha_2-3\alpha_2^2}}\alpha_2^{\frac{4\alpha_1^2+5\alpha_1\alpha_2+\alpha_2^2}{4\alpha_1^2+2\alpha_1\alpha_2-6\alpha_2^2}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1+\alpha_2)}{(6\alpha_2+4\alpha_1)}}}{(\gamma+\beta\alpha_1)^{\frac{5}{2}\alpha_1\frac{\alpha_1+\alpha_2}{2\alpha_1^2+\alpha_1\alpha_2-3\alpha_2^2}}k^{\frac{(\alpha_1+\alpha_2)}{2\alpha_1+3\alpha_2}}} \lesseqgtr \phi^{5\alpha_2\frac{\alpha_1+\alpha_2}{2\alpha_1^2+\alpha_1\alpha_2-3\alpha_2^2}}$$

Multiplying the power by $(2\alpha_1^2+\alpha_1\alpha_2-3\alpha_2^2)$ on both sides:

$$\frac{\beta^{\frac{5}{2}\alpha_1(\alpha_1+\alpha_2)}(1-\alpha_1)^{\alpha_1^2+5\alpha_1\alpha_2+4\alpha_2^2}\alpha_2^{2\alpha_1^2+\frac{5}{2}\alpha_1\alpha_2+\frac{1}{2}\alpha_2^2}(1-\alpha_1-\alpha_2)^{\frac{1}{2}\alpha_1^2-\frac{1}{2}\alpha_2^2}}{(\gamma+\beta\alpha_1)^{\frac{5}{2}\alpha_1(\alpha_1+\alpha_2)}k^{\alpha_1^2-\alpha_2^2}} \lesseqgtr \phi^{5\alpha_2(\alpha_1+\alpha_2)}$$

Dividing the power by $[5\alpha_2(\alpha_1+\alpha_2)]$:

$$\frac{\beta^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}(1-\alpha_1)^{\frac{(\alpha_1+4\alpha_2)}{5\alpha_2}}\alpha_2^{\frac{(4\alpha_1+\alpha_2)}{10\alpha_2}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1-\alpha_2)}{10\alpha_2}}}{(\gamma+\beta\alpha_1)^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}k^{\frac{\alpha_1^2-\alpha_2^2}{5\alpha_2(\alpha_1+\alpha_2)}}} \lesseqgtr \phi$$

$$\phi > \frac{\beta^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}(1-\alpha_1)^{\frac{(\alpha_1+4\alpha_2)}{5\alpha_2}}\alpha_2^{\frac{(4\alpha_1+\alpha_2)}{10\alpha_2}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1-\alpha_2)}{10\alpha_2}}}{(\gamma+\beta\alpha_1)^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}k^{\frac{\alpha_1^2-\alpha_2^2}{5\alpha_2(\alpha_1+\alpha_2)}}}$$

$$\phi > \frac{\beta^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}\left(1-\alpha_1\right)^{\frac{(\alpha_1+4\alpha_2)}{5\alpha_2}}\alpha_2^{\frac{(4\alpha_1+\alpha_2)}{10\alpha_2}}\left(1-\alpha_1-\alpha_2\right)^{\frac{(\alpha_1-\alpha_2)}{10\alpha_2}}}{\left(\gamma+\beta\alpha_1\right)^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}k^{\frac{\alpha_1^2-\alpha_2^2}{5\alpha_2(\alpha_1+\alpha_2)}}} \equiv \omega_A \tag{E.8}$$

Now we need to see if inequality (III) $\lesseqgtr$ Inequality (IV):

$$\left[\frac{\phi^{(6\alpha_2+4\alpha_1)}\alpha_2^{(2\alpha_2+\alpha_1)}\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}(1-\alpha_1)^{(4\alpha_2+2\alpha_1)}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}}\right]^{\frac{1}{(5\alpha_2+3\alpha_1)}}\left(\frac{1}{k}\right) \lesseqgtr \left[\frac{k^{(\alpha_2-\alpha_1)}(1-\alpha_1)^{2\alpha_2}}{\alpha_2^{\alpha_2}\phi^{(\alpha_2-\alpha_1)}}\left(\frac{\beta\alpha_2}{\gamma+\beta\alpha_1}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1-\alpha_2}}$$

$$\left[\frac{\phi^{\frac{(6\alpha_2+4\alpha_1)}{5\alpha_2+3\alpha_1}}\alpha_2^{\frac{(2\alpha_2+\alpha_1)}{5\alpha_2+3\alpha_1}}\alpha_2^{\frac{\alpha_2}{5\alpha_2+3\alpha_1}}(\gamma+\beta\alpha_1)^{\frac{\alpha_1}{(5\alpha_2+3\alpha_1)}}}{\beta^{\frac{\alpha_1}{(5\alpha_2+3\alpha_1)}}(1-\alpha_1)^{\frac{(4\alpha_2+2\alpha_1)}{(5\alpha_2+3\alpha_1)}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1+\alpha_2)}{(5\alpha_2+3\alpha_1)}}}\right]\left(\frac{1}{k}\right) \lesseqgtr \left[\frac{k^{\frac{(\alpha_2-\alpha_1)}{\alpha_1-\alpha_2}}(1-\alpha_1)^{\frac{2\alpha_2}{\alpha_1-\alpha_2}}}{\alpha_2^{\frac{\alpha_2}{\alpha_1-\alpha_2}}\phi^{\frac{(\alpha_2-\alpha_1)}{\alpha_1-\alpha_2}}}\left(\frac{\beta\alpha_2}{\gamma+\beta\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}\right]$$

$$\phi^{\frac{(6\alpha_2+4\alpha_1)}{5\alpha_2+3\alpha_1}}\phi^{\frac{(\alpha_2-\alpha_1)}{\alpha_1-\alpha_2}} \lesseqgtr$$

$$k\left(\frac{\beta\alpha_2}{\gamma+\beta\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}\frac{k^{\frac{(\alpha_2-\alpha_1)}{\alpha_1-\alpha_2}}(1-\alpha_1)^{\frac{2\alpha_2}{\alpha_1-\alpha_2}}\beta^{\frac{\alpha_1}{(5\alpha_2+3\alpha_1)}}(1-\alpha_1)^{\frac{(4\alpha_2+2\alpha_1)}{(5\alpha_2+3\alpha_1)}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1+\alpha_2)}{(5\alpha_2+3\alpha_1)}}}{\alpha_2^{\frac{(2\alpha_2+\alpha_1)}{5\alpha_2+3\alpha_1}}\alpha_2^{\frac{\alpha_2}{5\alpha_2+3\alpha_1}}(\gamma+\beta\alpha_1)^{\frac{\alpha_1}{(5\alpha_2+3\alpha_1)}}\alpha_2^{\frac{\alpha_2}{\alpha_1-\alpha_2}}}$$

$$\phi^{\frac{(\alpha_1+\alpha_2)}{3\alpha_1+5\alpha_2}} \lesseqgtr \frac{\beta^{4\alpha_1\frac{\alpha_1+\alpha_2}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}\alpha_2^{\frac{2(\alpha_1+\alpha_2)}{3\alpha_1+5\alpha_2}}(1-\alpha_1)^{2\frac{\alpha_1^2+4\alpha_1\alpha_2+3\alpha_2^2}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1+\alpha_2)}{(5\alpha_2+3\alpha_1)}}}{(\gamma+\beta\alpha_1)^{4\alpha_1\frac{\alpha_1+\alpha_2}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}}$$

Multiplying the power by $(3\alpha_1 + 5\alpha_2)$ :

$$\phi^{(\alpha_1+\alpha_2)} \lesseqgtr \frac{\beta^{4\alpha_1\frac{(3\alpha_1+5\alpha_2)(\alpha_1+\alpha_2)}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}\alpha_2^{2(\alpha_1+\alpha_2)}(1-\alpha_1)^{2\frac{\left(\alpha_1^2+4\alpha_1\alpha_2+3\alpha_2^2\right)(3\alpha_1+5\alpha_2)}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}(1-\alpha_1-\alpha_2)^{(\alpha_1+\alpha_2)}}{(\gamma+\beta\alpha_1)^{4\alpha_1\frac{(3\alpha_1+5\alpha_2)(\alpha_1+\alpha_2)}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}}$$

Dividing the power by $(\alpha_1 + \alpha_2)$ :

$$\phi \lesseqgtr \frac{\beta^{4\alpha_1\frac{(3\alpha_1+5\alpha_2)}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}\alpha_2^2(1-\alpha_1)^{2\frac{\left(\alpha_1^2+4\alpha_1\alpha_2+3\alpha_2^2\right)(3\alpha_1+5\alpha_2)}{(\alpha_1+\alpha_2)\left(3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2\right)}}(1-\alpha_1-\alpha_2)}{(\gamma+\beta\alpha_1)^{4\alpha_1\frac{(3\alpha_1+5\alpha_2)}{3\alpha_1^2+2\alpha_1\alpha_2-5\alpha_2^2}}}$$

$$\phi \lesseqgtr \frac{\beta^{\frac{4\alpha_1}{\alpha_1-\alpha_2}}\alpha_2^2(1-\alpha_1)^{\frac{2\alpha_1+6\alpha_2}{\alpha_1-\alpha_2}}(1-\alpha_1-\alpha_2)}{(\gamma+\beta\alpha_1)^{\frac{4\alpha_1}{\alpha_1-\alpha_2}}}$$

$$\phi < \left[\left(\frac{\beta}{(\gamma+\beta\alpha_1)}\right)^{4\alpha_1}(1-\alpha_1)^{2\alpha_1+6\alpha_2}\right]^{\frac{1}{\alpha_1-\alpha_2}} \alpha_2^2\,(1-\alpha_1-\alpha_2) \equiv \omega_B \qquad \text{(E.9)}$$

Now we see whether $\omega_A \lesseqgtr \omega_B$

$$\frac{\beta^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}(1-\alpha_1)^{\frac{(\alpha_1+4\alpha_2)}{5\alpha_2}}\alpha_2^{\frac{(4\alpha_1+\alpha_2)}{10\alpha_2}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1-\alpha_2)}{10\alpha_2}}}{(\gamma+\beta\alpha_1)^{\frac{1}{2}\frac{\alpha_1}{\alpha_2}}k^{\frac{\alpha_1^2-\alpha_2^2}{5\alpha_2(\alpha_1+\alpha_2)}}} \lesseqgtr \frac{\beta^{\frac{4\alpha_1}{\alpha_1-\alpha_2}}\alpha_2^2(1-\alpha_1)^{\frac{2\alpha_1+6\alpha_2}{\alpha_1-\alpha_2}}(1-\alpha_1-\alpha_2)}{(\gamma+\beta\alpha_1)^{\frac{4\alpha_1}{\alpha_1-\alpha_2}}}$$

$$\frac{\beta^{\frac{1}{2}\frac{\alpha_1}{\alpha_2(\alpha_1-\alpha_2)}(\alpha_1-9\alpha_2)}\alpha_2^{\frac{(4\alpha_1-19\alpha_2)}{10\alpha_2}}}{(\gamma+\beta\alpha_1)^{\frac{1}{2}\frac{\alpha_1}{\alpha_2(\alpha_1-\alpha_2)}(\alpha_1-9\alpha_2)}(1-\alpha_1)^{\frac{\left(7\alpha_1\alpha_2+34\alpha_2^2-\alpha_1^2\right)}{5\alpha_2(\alpha_1-\alpha_2)}}(1-\alpha_1-\alpha_2)^{\frac{(\alpha_1-11\alpha_2)}{10\alpha_2}}} \lesseqgtr k^{\frac{\alpha_1^2-\alpha_2^2}{5\alpha_2(\alpha_1+\alpha_2)}}$$

Multiplying the power by $[5\alpha_2\,(\alpha_1+\alpha_2)]$ and dividing it by $(\alpha_1^2-\alpha_2^2)$ :

$$k < \left[\frac{\beta^{\frac{\left(5\alpha_1^3-40\alpha_1^2\alpha_2-45\alpha_1\alpha_2^2\right)}{(2\alpha_1-2\alpha_2)}}\alpha_2^{\frac{1}{2}(4\alpha_1-19\alpha_2)(\alpha_1+\alpha_2)}}{(\gamma+\beta\alpha_1)^{\frac{5}{2}\alpha_1\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}(\alpha_1-9\alpha_2)}(1-\alpha_1)^{\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}\left(34\alpha_2^2-\alpha_1^2+7\alpha_1\alpha_2\right)}(1-\alpha_1-\alpha_2)^{\frac{1}{2}(\alpha_1+\alpha_2)(\alpha_1-11\alpha_2)}}\right]^{\frac{1}{\alpha_1^2-\alpha_2^2}} \equiv \tilde{k} \qquad \text{(E.10)}$$

Therefore, $\omega_B > \omega_A$ when $k < \tilde{k}$.

# 12 Appendix F

This Appendix provides the calculation of the steady-state values of $e^*, n^*$ and $N^*$. In order to derive these values, we would substitute $h_t$ with E.1, E.2 and E.3.

**Time dedicated to Education:**

(i) In first regime steady-state value of $e^*$ would be a constant value equivalent to zero:

$$e^*_{sL} = 0 \tag{F.1}$$

(ii) In second regime substitute E.2 in A.4:

$$e^*_2 = \frac{(h_t+k)[\alpha_2\phi-\theta(1-\alpha_1)]-k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}$$

$$e^*_2 = \frac{\alpha_2\phi-\theta(1-\alpha_1)}{(1-\alpha_1-\alpha_2)} - \frac{k\alpha_2}{(h_t+k)(1-\alpha_1-\alpha_2)}$$

$$e^*_2\,|_{h_t=h_m} = \frac{\alpha_2\phi-\theta(1-\alpha_1)}{(1-\alpha_1-\alpha_2)} - \frac{k\alpha_2}{\left(\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}+k\right)(1-\alpha_1-\alpha_2)}$$

$$e^*_{sm} = \left[\alpha_2\phi-\theta\left(1-\alpha_1\right)\right]\left(1-\alpha_1-\alpha_2\right)^{-1} - \frac{k\alpha_2(1-\alpha_1-\alpha_2)^{-1}}{\left[\frac{(\phi-\theta)-k(1+\theta-\phi)}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}+k}$$

$$e^*_{sm} = \left[\alpha_2\phi-\theta\left(1-\alpha_1\right)\right]\left(1-\alpha_1-\alpha_2\right)^{-1} -$$
$$\left[k\alpha_2\left(1-\alpha_1-\alpha_2\right)^{-1}\right]\left\{\left[\frac{(\phi-\theta)-k(1+\theta-\phi)}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}+k\right\}^{-1}$$

Taking logs in order to simplify the above expression

$$\log e^*_{sm} =$$
$$\log\alpha_2+\log\phi-\log\theta-\log\left(1-\alpha_1\right)-\log\left(1-\alpha_1-\alpha_2\right)+\log k+\log\alpha_2+\log\left(1-\alpha_1-\alpha_2\right)$$
$$-\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\log\left(\phi-\theta\right)+\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\log k+\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\log\left(1+\theta-\phi\right)+\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\log\left(1-\alpha_1-\alpha_2\right)$$
$$-\left(\tfrac{\alpha_2}{2\alpha_2+\alpha_1}\right)\log\alpha_2-\left(\tfrac{\alpha_1}{2\alpha_2+\alpha_1}\right)\log\left(\gamma+\beta\alpha_1\right)+\left(\tfrac{\alpha_1}{2\alpha_2+\alpha_1}\right)\log\beta+\left(\tfrac{\alpha_2}{2\alpha_2+\alpha_1}\right)\log k-\log k$$
$$\log e^*_{sm} = \left(\tfrac{2\alpha_1+3\alpha_2}{\alpha_1+2\alpha_2}\right)\log\alpha_2-\left(\tfrac{\alpha_1}{\alpha_1+2\alpha_2}\right)\log\phi+\left(\tfrac{\alpha_1}{\alpha_1+2\alpha_2}\right)\log\theta-\log\left(1-\alpha_1\right)$$
$$+\tfrac{\alpha_1+\alpha_2}{\alpha_1+2\alpha_2}\log\left(1-\alpha_1-\alpha_2\right)+\log k-\left(\tfrac{\alpha_1}{\alpha_1+2\alpha_2}\right)\log\left(\gamma+\beta\alpha_1\right)+\left(\tfrac{\alpha_1}{\alpha_1+2\alpha_2}\right)\log\beta$$

Taking exponentials:

$$e^*_{sm} = \left(\frac{\alpha_2^{2\alpha_1+3\alpha_2}\theta^{\alpha_1}\left(1-\alpha_1-\alpha_2\right)^{\alpha_1+\alpha_2}\beta^{\alpha_1}}{\phi^{\alpha_1}\left(\gamma+\beta\alpha_1\right)^{\alpha_1}}\right)^{\frac{1}{\alpha_1+2\alpha_2}}\left(\frac{k}{(1-\alpha_1)}\right) \tag{F.2}$$

(iii) In third regime steady-state value of $e^*$ would be a constant value equivalent to the maximum value attained when there is zero child labor i.e.

$$e^*_{sH} = 1 - \phi \tag{F.3}$$

**Child Nutrition:**

(i) In first regime, substitute E.1 in A.11:

$$N_1^* = \tfrac{\theta(\gamma+\beta\alpha_1)(h_t+k)}{\beta\alpha_2}$$
$$N_1^* = \tfrac{\theta(\gamma+\beta\alpha_1)}{\beta\alpha_2}(h_t+k)$$
$$N_1^*|_{h_t=h_L} = \tfrac{\theta(\gamma+\beta\alpha_1)}{\beta\alpha_2}\left\{\left[\tfrac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}\theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}}+k\right\}$$
$$N_{sL}^* = \frac{\theta^{\frac{(\alpha_1+2\alpha_2)}{\alpha_2}}k^{\frac{\alpha_1}{\alpha_2}}(\gamma+\beta\alpha_1)^{\frac{(\alpha_1+\alpha_2)}{\alpha_2}}}{(\beta\alpha_2)^{\frac{(\alpha_1+\alpha_2)}{\alpha_2}}}+\tfrac{\theta(\gamma+\beta\alpha_1)}{\beta\alpha_2}k$$

Taking logs in order to simplify the above expression:

$$\log N_{sL}^* = \left(\tfrac{\alpha_1+2\alpha_2}{\alpha_2}\right)\log\theta+\left(\tfrac{\alpha_1}{\alpha_2}\right)\log k+\left(\tfrac{\alpha_1+\alpha_2}{\alpha_2}\right)\log(\gamma+\beta\alpha_1)-\left(\tfrac{\alpha_1+\alpha_2}{\alpha_2}\right)\log\beta\alpha_2+$$
$$\log\theta+\log k+\log(\gamma+\beta\alpha_1)-\log\beta\alpha_2$$
$$\log N_{sL}^* = \left(\tfrac{\alpha_1+3\alpha_2}{\alpha_2}\right)\log\theta+\left(\tfrac{\alpha_1+\alpha_2}{\alpha_2}\right)\log k+\left(\tfrac{\alpha_1+2\alpha_2}{\alpha_2}\right)\log(\gamma+\beta\alpha_1)-\left(\tfrac{\alpha_1+2\alpha_2}{\alpha_2}\right)\log\beta\alpha_2$$

Taking exponentials:

$$N_{sL}^* = \frac{\theta^{\frac{\alpha_1+3\alpha_2}{\alpha_2}}k^{\frac{\alpha_1+\alpha_2}{\alpha_2}}(\gamma+\beta\alpha_1)^{\frac{\alpha_1+2\alpha_2}{\alpha_2}}}{(\beta\alpha_2)^{\frac{\alpha_1+2\alpha_2}{\alpha_2}}}$$

$$N_{sL}^* = \left[\frac{\theta^{(\alpha_1+3\alpha_2)}k^{(\alpha_1+\alpha_2)}(\gamma+\beta\alpha_1)^{(\alpha_1+2\alpha_2)}}{(\beta\alpha_2)^{(\alpha_1+2\alpha_2)}}\right]^{\frac{1}{\alpha_2}} \tag{F.4}$$

(ii) In second regime, substitute E.2 in A.13:

$$N_2^* = \tfrac{(\gamma+\beta\alpha_1)[(h_t+k)(\phi-\theta)-k]}{\beta(1-\alpha_1-\alpha_2)}$$
$$N_2^* = \tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}[(h_t+k)(\phi-\theta)-k]$$
$$N_2^* = \tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(h_t+k)(\phi-\theta)-\tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}k$$
$$N_2^* = \tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(\phi-\theta)h_t+\tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(\phi-\theta)k-\tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}k$$

$$N_2^* = \tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(\phi-\theta)h_t-\tfrac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(1-\phi+\theta)k$$

$$N_2^* \left|_{h_t=h_m}\right. =$$

$$\frac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(\phi-\theta)\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}-\frac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(1-\phi+\theta)\,k$$

$$N_{sm}^* =$$

$$\frac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(\phi-\theta)\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}-\frac{(\gamma+\beta\alpha_1)}{\beta(1-\alpha_1-\alpha_2)}(1-\phi+\theta)\,k$$

Taking logs to simplify the above expression:

In the above expression assume $x = \frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}$ and $y = \frac{1}{2\alpha_2+\alpha_1}$

$$\log N_{sm}^* = \log(\gamma+\beta\alpha_1) + \log\phi - \log\theta - \log\beta - \log(1-\alpha_1-\alpha_2) + x\log\phi - x\log k -$$

$$x\log(1-\phi) - x\log\theta$$

$$-x\log(1+k) - x\log(1-\alpha_1-\alpha_2) + y\alpha_2\log\alpha_2 + y\alpha_1\log(\gamma+\beta\alpha_1) - y\alpha_1\log\beta -$$

$$y\alpha_2\log k - \log(\gamma+\beta\alpha_1)$$

$$-\log(1-\phi+\theta) - \log k + \log\beta + \log(1-\alpha_1-\alpha_2)$$

$$\log N_{sm}^* = y\alpha_1\log(\gamma+\beta\alpha_1) + (2+2x)\log\phi - (2+x)\log\theta - y\alpha_1\log\beta -$$

$$x\log(1-\alpha_1-\alpha_2) - (2x+y\alpha_2)\log k + y\alpha_2\log\alpha_2$$

Taking exponentials and substituting back the values of $x$ and $y$:

$$N_{sm}^* = \frac{(\gamma+\beta\alpha_1)^{\frac{\alpha_1}{2\alpha_2+\alpha_1}}\phi^{\frac{4\alpha_1+6\alpha_2}{\alpha_1+2\alpha_2}}\alpha_2^{\frac{\alpha_2}{2\alpha_2+\alpha_1}}}{\theta^{\frac{3\alpha_1+5\alpha_2}{\alpha_1+2\alpha_2}}\beta^{\frac{\alpha_1}{2\alpha_2+\alpha_1}}k^{\frac{2\alpha_1+3\alpha_2}{\alpha_1+2\alpha_2}}(1-\alpha_1-\alpha_2)^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}}$$

$$N_{sm}^* = \left[\frac{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(4\alpha_1+6\alpha_2)}\alpha_2^{\alpha_2}}{\theta^{3\alpha_1+5\alpha_2}\beta^{\alpha_1}k^{2\alpha_1+3\alpha_2}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}\right]^{\frac{1}{\alpha_1+2\alpha_2}} \tag{F.5}$$

(iii) In the third regime substitute E.3 in A.14:

$$N_3^* = \frac{(\gamma+\beta\alpha_1)(h_t+k)(1+\theta-\phi)}{\beta\alpha_2}$$

$$N_3^* = \frac{(\gamma+\beta\alpha_1)(1+\theta-\phi)}{\beta\alpha_2}(h_t+k)$$

$$N_3^* \left|_{h_t=h_H}\right. = \frac{(\gamma+\beta\alpha_1)(1+\theta-\phi)}{\beta\alpha_2}\left\{\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}+k\right\}$$

$$N_{sH}^* = \frac{(\gamma+\beta\alpha_1)(1+\theta-\phi)}{\beta\alpha_2}\left\{\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}+k\right\}$$

Taking logs in order to simplify the above expression:

$$\log N_{sH}^{*} = \log(\gamma+\beta\alpha_1) + \log(1+\theta-\phi) - \log(\beta\alpha_2) + \log k + \left(\frac{\alpha_1}{\alpha_2}\right)\log k +$$

$$\left(\frac{\alpha_1}{\alpha_2}\right)\log(\gamma+\beta\alpha_1) - \left(\frac{\alpha_1}{\alpha_2}\right)\log(\beta\alpha_2)$$

$$+\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)\log(1+\theta-\phi)$$

$$\log N_{sH}^{*} =$$

$$\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)\log(\gamma+\beta\alpha_1)+\left(\frac{\alpha_1+2\alpha_2}{\alpha_2}\right)\log\theta-\left(\frac{\alpha_1+2\alpha_2}{\alpha_2}\right)\log\phi-\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)\log(\beta\alpha_2)+\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)\log k$$

Taking exponentials:

$$N_{sH}^{*} = \left[\frac{\theta^{\alpha_1+2\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1+\alpha_2}\, k^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1+\alpha_2}\,\phi^{\alpha_1+2\alpha_2}}\right]^{\frac{1}{\alpha_2}} \tag{F.6}$$

**Number of Children/Fertility:**

(i) In first regime substitute E.1 in A.6:

$$n_1^{*} = \frac{\beta\alpha_2 h_t}{(h_t+k)[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$

$$n_1^{*}|_{h_t=h_L} = \frac{\beta\alpha_2\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}\theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}}}{\left\{\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}\theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}}+k\right\}[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$

$$n_{sL}^{*} = \frac{\theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}}[(\gamma+\beta\alpha_1)k]^{\frac{\alpha_1}{\alpha_2}}(\beta\alpha_2)^{-\frac{\alpha_1-\alpha_2}{\alpha_2}}}{\left\{\theta^{\frac{\alpha_1+\alpha_2}{\alpha_2}}[(\gamma+\beta\alpha_1)k]^{\frac{\alpha_1}{\alpha_2}}(\beta\alpha_2)^{-\frac{\alpha_1}{\alpha_2}}+k\right\}[\theta(\gamma+\beta\alpha_1)+\beta\alpha_2\phi]-\beta\alpha_2 k}$$

Taking logs to simplify the above expression:

$$\log n_{sL}^{*} = \left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)\log\theta + \left(\frac{\alpha_1}{\alpha_2}\right)\log k + \left(\frac{\alpha_1}{\alpha_2}\right)\log(\gamma+\beta\alpha_1) - \left(\frac{\alpha_1-\alpha_2}{\alpha_2}\right)\log(\beta\alpha_2) -$$

$$\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)\log\theta - \left(\frac{\alpha_1}{\alpha_2}\right)\log k$$

$$-\left(\frac{\alpha_1}{\alpha_2}\right)\log(\gamma+\beta\alpha_1) + \left(\frac{\alpha_1}{\alpha_2}\right)\log(\beta\alpha_2) - \log k - \log\theta - \log(\gamma+\beta\alpha_1) - \log(\beta\alpha_2) -$$

$$\log\phi + \log(\beta\alpha_2) + \log k$$

$$\log n_{sL}^{*} = -\log\theta - \log(\gamma+\beta\alpha_1) + \log(\beta\alpha_2) - \log\phi$$

Taking exponentials:

$$n_{sL}^{*} = \frac{(\beta\alpha_2)}{\phi\theta(\gamma+\beta\alpha_1)} \tag{F.7}$$

(ii) In second regime substitute E.2 in A.8:

$$n_2^{*} = \frac{\beta h_t(1-\alpha_1-\alpha_2)}{(\gamma+\beta)[(h_t+k)(\phi-\theta)-k]}$$

$$n_2^*|_{h_t=h_m} = \frac{\beta(1-\alpha_1-\alpha_2)\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}}{(\gamma+\beta)\left\{\left(\left[\frac{[\phi-k(1-\phi)-\theta(1+k)]}{1-\alpha_1-\alpha_2}\right]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\frac{\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}}{\beta^{\alpha_1}k^{\alpha_2}}\right]^{\frac{1}{2\alpha_2+\alpha_1}}+k\right)(\phi-\theta)-k\right\}}$$

$$n_{sm}^* = \frac{\beta^{\frac{2\alpha_2}{\alpha_1+2\alpha_2}}(1-\alpha_1-\alpha_2)^{\frac{\alpha_2}{\alpha_1+2\alpha_2}}[\phi-k(1-\phi)-\theta(1+k)]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\right]^{\frac{1}{2\alpha_2+\alpha_1}}k^{-\frac{\alpha_2}{2\alpha_2+\alpha_1}}}{(\gamma+\beta)\left\{\left((1-\alpha_1-\alpha_2)^{-\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}[\phi-k(1-\phi)-\theta(1+k)]^{\frac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}}\left[\alpha_2^{\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1}\right]^{\frac{1}{2\alpha_2+\alpha_1}}(\beta^{\alpha_1}k^{\alpha_2})^{-\frac{1}{2\alpha_2+\alpha_1}}+k\right)(\phi-\theta)-k\right\}}$$

Taking logs to simplify the above expression:

$$\log n_{sm}^* = \left(\tfrac{2\alpha_2}{\alpha_1+2\alpha_2}\right)\log\beta + \left(\tfrac{\alpha_2}{\alpha_1+2\alpha_2}\right)\log(1-\alpha_1-\alpha_2) + \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log\phi - \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log k + \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log\phi$$
$$- \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log\theta - \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log(1+k) + \left(\tfrac{\alpha_2}{2\alpha_2+\alpha_1}\right)\log\alpha_2 + \left(\tfrac{\alpha_1}{2\alpha_2+\alpha_1}\right)\log(\gamma+\beta\alpha_1) - \left(\tfrac{\alpha_2}{2\alpha_2+\alpha_1}\right)\log k - \log(\gamma+\beta)$$
$$+ \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log(1-\alpha_1-\alpha_2) - \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log\phi + \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log k - \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log(1-\phi) + \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log\theta - \left(\tfrac{\alpha_1+\alpha_2}{2\alpha_2+\alpha_1}\right)\log(1+k)$$
$$- \left(\tfrac{\alpha_2}{2\alpha_2+\alpha_1}\right)\log\alpha_2 - \left(\tfrac{\alpha_1}{2\alpha_2+\alpha_1}\right)\log(\gamma+\beta\alpha_1) + \left(\tfrac{\alpha_1}{2\alpha_2+\alpha_1}\right)\log\beta + \left(\tfrac{\alpha_2}{2\alpha_2+\alpha_1}\right)\log k - \log k - \log(\phi-\theta) + \log k$$

$$\log n_{sm}^* = \log\beta + \log(1-\alpha_1-\alpha_2) + \left(\tfrac{\alpha_1}{\alpha_1+2\alpha_2}\right)\log\phi - \left(2\tfrac{(\alpha_1+\alpha_2)}{\alpha_1+2\alpha_2}\right)\log k + \log\theta + \log(\gamma+\beta\alpha_1) - \log(\gamma+\beta)$$

Taking exponentials:

$$n_{sm}^* = \frac{(\gamma+\beta\alpha_1)\,\theta\beta\,(1-\alpha_1-\alpha_2)\,\phi^{\frac{\alpha_1}{\alpha_1+2\alpha_2}}}{k^{\left(2\frac{(\alpha_1+\alpha_2)}{\alpha_1+2\alpha_2}\right)}(\gamma+\beta)} \tag{F.8}$$

(iii) In the third regime substitute E.3 in A.9:

$$n_3^* = \frac{\beta\alpha_2 h_t}{(h_t+k)[(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2]-\beta\alpha_2 k}$$

$$n_3^* = \frac{\beta\alpha_2 h_t}{(h_t+k)(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2 h_t}$$

$$n_3^*|_{h_t=h_H} = \frac{\beta\alpha_2\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}}{\left(\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}+k\right)(1-\phi+\theta)(\gamma+\beta\alpha_1)+\beta\alpha_2\left[\frac{(\gamma+\beta\alpha_1)k}{(\beta\alpha_2)}\right]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}}$$

$$n_{s_H}^* = \frac{(\beta\alpha_2)^{-\frac{(\alpha_1-\alpha_2)}{\alpha_2}}[(\gamma+\beta\alpha_1)k]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}}{\left[(\beta\alpha_2)^{-\frac{\alpha_1}{\alpha_2}}[(\gamma+\beta\alpha_1)k]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}+k\right](1-\phi+\theta)(\gamma+\beta\alpha_1)+(\beta\alpha_2)^{-\frac{(\alpha_1-\alpha_2)}{\alpha_2}}[(\gamma+\beta\alpha_1)k]^{\frac{\alpha_1}{\alpha_2}}(1+\theta-\phi)^{\frac{\alpha_1+\alpha_2}{\alpha_2}}}$$

Taking logs to simplify the above expression:

$$\log n^*_{s_H} =$$
$$-\left(\tfrac{(\alpha_1-\alpha_2)}{\alpha_2}\right)\log(\beta\alpha_2)+\left(\tfrac{\alpha_1}{\alpha_2}\right)\log k+\left(\tfrac{\alpha_1}{\alpha_2}\right)\log(\gamma+\beta\alpha_1)+\left(\tfrac{\alpha_1+\alpha_2}{\alpha_2}\right)\log(1+\theta-\phi)$$
$$+\left(\tfrac{\alpha_1}{\alpha_2}\right)\log(\beta\alpha_2)-\left(\tfrac{\alpha_1}{\alpha_2}\right)\log k-\left(\tfrac{\alpha_1}{\alpha_2}\right)\log(\gamma+\beta\alpha_1)-\left(\tfrac{\alpha_1+\alpha_2}{\alpha_2}\right)\log(1+\theta-\phi)-\log k$$
$$-\log(1-\phi+\theta)-\log(\gamma+\beta\alpha_1)-\left(\tfrac{(\alpha_1-\alpha_2)}{\alpha_2}\right)\log(\beta\alpha_2)-\left(\tfrac{\alpha_1}{\alpha_2}\right)\log k$$
$$-\left(\tfrac{\alpha_1}{\alpha_2}\right)\log(\gamma+\beta\alpha_1)-\left(\tfrac{\alpha_1+\alpha_2}{\alpha_2}\right)\log(1+\theta-\phi)$$

$$\log n^*_{s_H} =$$
$$-\tfrac{(\alpha_1-2\alpha_2)}{\alpha_2}\log(\beta\alpha_2)-\tfrac{(\alpha_1+\alpha_2)}{\alpha_2}\log k-\tfrac{(\alpha_1+\alpha_2)}{\alpha_2}\log(\gamma+\beta\alpha_1)-\tfrac{(\alpha_1+2\alpha_2)}{\alpha_2}\log(1-\phi+\theta)$$

Taking exponentials:

$$n^*_{s_H} = \left[\frac{1}{(1-\phi+\theta)^{(\alpha_1+2\alpha_2)}\,(\gamma+\beta\alpha_1)^{(\alpha_1+\alpha_2)}\,(\beta\alpha_2)^{(\alpha_1-2\alpha_2)}\,k^{(\alpha_1+\alpha_2)}}\right]^{\frac{1}{\alpha_2}} \quad \text{(F.9)}$$

# 13 Appendix G

In this appendix a detailed comparative static analysis will be presented of the steady-state values of $e^*, n^*$ and $N^*$ with respect to $k$ and $\theta$.

The optimal time devoted by a child in education in the steady-state $h_L$ and $h_m$ is 0 and $(1-\phi)$ respectively. Since both are not dependent on either $k$ or $\theta$, hence they both yield zero derivatives with respect to both.

Following is the optimal time devoted by a child in education in the steady-state $h_m$ :

$$e^*_{sm} = \left( \frac{\alpha_2^{2\alpha_1+3\alpha_2} \theta^{\alpha_1} (1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2} \beta^{\alpha_1}}{\phi^{\alpha_1} (\gamma+\beta\alpha_1)^{\alpha_1}} \right)^{\frac{1}{\alpha_1+2\alpha_2}} \left( \frac{k}{(1-\alpha_1)} \right)$$

Derivating $e^*_{sm}$ with respect to $k$ shows a positive impact of child labor wage on $e^*_{sm}$ :

$$\frac{\partial e^*_{sm}}{\partial k} = \left( \frac{\alpha_2^{2\alpha_1+3\alpha_2} \theta^{\alpha_1} (1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2} \beta^{\alpha_1}}{\phi^{\alpha_1} (\gamma+\beta\alpha_1)^{\alpha_1}} \right)^{\frac{1}{\alpha_1+2\alpha_2}} \left( \frac{1}{(1-\alpha_1)} \right)$$

$$\frac{\partial e^*_{sm}}{\partial k} > 0$$

Derivating $e^*_{sm}$ with respect to $\theta$ shows a positive impact of minimum skill level on $e^*_{sm}$ :

$$\frac{\partial e^*_{sm}}{\partial \theta} = \frac{\alpha_1}{\alpha_1+2\alpha_2} \theta^{-2\frac{\alpha_2}{\alpha_1+2\alpha_2}} \left( \frac{\alpha_2^{2\alpha_1+3\alpha_2} (1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2} \beta^{\alpha_1}}{\phi^{\alpha_1} (\gamma+\beta\alpha_1)^{\alpha_1}} \right)^{\frac{1}{\alpha_1+2\alpha_2}} \left( \frac{k}{(1-\alpha_1)} \right)$$

$$\frac{\partial e^*_{sm}}{\partial \theta} = \frac{1}{\theta^{2\frac{\alpha_2}{\alpha_1+2\alpha_2}}} \left( \frac{\alpha_1}{\alpha_1+2\alpha_2} \right) \left( \frac{\alpha_2^{2\alpha_1+3\alpha_2} (1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2} \beta^{\alpha_1}}{\phi^{\alpha_1} (\gamma+\beta\alpha_1)^{\alpha_1}} \right)^{\frac{1}{\alpha_1+2\alpha_2}} \left( \frac{k}{(1-\alpha_1)} \right)$$

$$\frac{\partial e^*_{sm}}{\partial \theta} > 0$$

Following is the optimal child nutrition in steady-state $h_L$ :

$$N^*_{sL} = \left[ \frac{\theta^{(\alpha_1+3\alpha_2)} k^{(\alpha_1+\alpha_2)} (\gamma+\beta\alpha_1)^{(\alpha_1+2\alpha_2)}}{(\beta\alpha_2)^{(\alpha_1+2\alpha_2)}} \right]^{\frac{1}{\alpha_2}}$$

Derivating $N^*_{sL}$ with respect to $k$ shows a positive impact of child labor wage on $N^*_{sL}$ :

$$\frac{\partial N^*_{sL}}{\partial k} = \frac{(\alpha_1+\alpha_2)}{\alpha_2} k^{\alpha_1+\alpha_2-1} \left[ \frac{\theta^{(\alpha_1+3\alpha_2)} (\gamma+\beta\alpha_1)^{(\alpha_1+2\alpha_2)}}{(\beta\alpha_2)^{(\alpha_1+2\alpha_2)}} \right]^{\frac{1}{\alpha_2}}$$

$$\frac{\partial N^*_{sL}}{\partial k} > 0$$

Derivating $N^*_{sL}$ with respect to $\theta$ shows a positive impact of minimum skill level on $N^*_{sL}$ :

$$\frac{\partial N^*_{sL}}{\partial \theta} = \left(\frac{\alpha_1+3\alpha_2}{\alpha_2}\right)\theta^{\alpha_1+3\alpha_2-1}\left[\frac{k^{(\alpha_1+\alpha_2)}(\gamma+\beta\alpha_1)^{(\alpha_1+2\alpha_2)}}{(\beta\alpha_2)^{(\alpha_1+2\alpha_2)}}\right]^{\frac{1}{\alpha_2}}$$

$$\frac{\partial N^*_{sL}}{\partial \theta} > 0$$

Following is the optimal child nutrition in steady-state $h_m$ :

$$N^*_{sm} = \left[\frac{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(4\alpha_1+6\alpha_2)}\alpha_2^{\alpha_2}}{\theta^{3\alpha_1+5\alpha_2}\beta^{\alpha_1}k^{2\alpha_1+3\alpha_2}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}\right]^{\frac{1}{\alpha_1+2\alpha_2}}$$

Derivating $N^*_{sm}$ with respect to $k$ shows a negative impact of child labor wage on $N^*_{sm}$ :

$$\frac{\partial N^*_{sm}}{\partial k} = -\frac{2\alpha_1+3\alpha_2}{\alpha_1+2\alpha_2}k^{\frac{\alpha_1+\alpha_2}{\alpha_1+2\alpha_2}}\left[\frac{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(4\alpha_1+6\alpha_2)}\alpha_2^{\alpha_2}}{\theta^{3\alpha_1+5\alpha_2}\beta^{\alpha_1}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}\right]^{\frac{1}{\alpha_1+2\alpha_2}}$$

$$\frac{N^*_{sm}}{\partial k} < 0$$

Derivating $N^*_{sm}$ with respect to $\theta$ shows a negative impact of minimum skills level on $N^*_{sm}$ :

$$\frac{\partial N^*_{sm}}{\partial \theta} = -\frac{3\alpha_1+5\alpha_2}{\alpha_1+2\alpha_2}\theta^{\frac{2\alpha_1+3\alpha_2}{\alpha_1+2\alpha_2}}\left[\frac{(\gamma+\beta\alpha_1)^{\alpha_1}\phi^{(4\alpha_1+6\alpha_2)}\alpha_2^{\alpha_2}}{\beta^{\alpha_1}k^{2\alpha_1+3\alpha_2}(1-\alpha_1-\alpha_2)^{\alpha_1+\alpha_2}}\right]^{\frac{1}{\alpha_1+2\alpha_2}}$$

$$\frac{\partial N^*_{sm}}{\partial \theta} < 0$$

Following is the optimal child nutrition in steady-state $h_H$ :

$$N^*_{sH} = \left[\frac{\theta^{\alpha_1+2\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1+\alpha_2}k^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1+\alpha_2}\phi^{\alpha_1+2\alpha_2}}\right]^{\frac{1}{\alpha_2}}$$

Derivating $N^*_{sH}$ with respect to $k$ shows a positive impact of child labor wage on $N^*_{sH}$ :

$$\frac{\partial N^*_{sH}}{\partial k} = \left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)k^{\frac{\alpha_1}{\alpha_2}}\left[\frac{\theta^{\alpha_1+2\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1+\alpha_2}\phi^{\alpha_1+2\alpha_2}}\right]^{\frac{1}{\alpha_2}}$$

$$\frac{\partial N^*_{sH}}{\partial k} > 0$$

Derivating $N^*_{sH}$with respect to $\theta$ shows a positive impact of minimum skills level on $N^*_{sH}$ :

$$\frac{\partial N^*_{sH}}{\partial \theta} = \left(\frac{\alpha_1+2\alpha_2}{\alpha_2}\right)\theta^{\frac{(\alpha_1+\alpha_2)}{\alpha_2}}\left[\frac{(\gamma+\beta\alpha_1)^{\alpha_1+\alpha_2}k^{\alpha_1+\alpha_2}}{(\beta\alpha_2)^{\alpha_1+\alpha_2}\phi^{\alpha_1+2\alpha_2}}\right]^{\frac{1}{\alpha_2}}$$

$$\frac{\partial N^*_{sH}}{\partial \theta} > 0$$

Following is the optimal fertility in steady-state $h_L$ :

$$n^*_{sL} = \frac{(\beta\alpha_2)}{\phi\theta(\gamma+\beta\alpha_1)}$$

Derivating $n^*_{sL}$ with respect to $k$ shows a that child labor wage has no impact on $n^*_{sL}$ since $n^*_{sL}$ is not a function of $k$ :

$$\frac{\partial n^*_{sL}}{\partial k} = 0$$

Derivating $n^*_{sL}$ with respect to $\theta$ shows a negative impact of minimum skills level on $n^*_{sL}$ :

$$\frac{\partial n^*_{sL}}{\partial \theta} = -\left(\frac{1}{\theta^2}\right)\left(\frac{(\beta\alpha_2)}{\phi(\gamma+\beta\alpha_1)}\right)$$
$$\frac{\partial n^*_{sL}}{\partial \theta} < 0$$

Following is the optimal fertility in steady-state $h_m$ :

$$n^*_{sm} = \frac{(\gamma+\beta\alpha_1)\theta\beta(1-\alpha_1-\alpha_2)\phi^{\frac{\alpha_1}{\alpha_1+2\alpha_2}}}{k^{\left(2\frac{(\alpha_1+\alpha_2)}{\alpha_1+2\alpha_2}\right)}(\gamma+\beta)}$$

Derivating $n^*_{sm}$ with respect to $k$ shows a negative impact of child labor wage on $n^*_{sm}$ :

$$\frac{\partial n^*_{sm}}{\partial k} = -\left(2\frac{(\alpha_1+\alpha_2)}{\alpha_1+2\alpha_2}\right)k^{\frac{\alpha_1}{\alpha_1+2\alpha_2}}\left(\frac{(\gamma+\beta\alpha_1)\theta\beta(1-\alpha_1-\alpha_2)\phi^{\frac{\alpha_1}{\alpha_1+2\alpha_2}}}{(\gamma+\beta)}\right)$$
$$\frac{\partial n^*_{sm}}{\partial k} < 0$$

Derivating $n^*_{sm}$ with respect to $\theta$ shows a positive impact of minimum skills level on $n^*_{sL}$ :

$$\frac{\partial n^*_{sm}}{\partial \theta} = \frac{(\gamma+\beta\alpha_1)\beta(1-\alpha_1-\alpha_2)\phi^{\frac{\alpha_1}{\alpha_1+2\alpha_2}}}{k^{\left(2\frac{(\alpha_1+\alpha_2)}{\alpha_1+2\alpha_2}\right)}(\gamma+\beta)}$$
$$\frac{\partial n^*_{sm}}{\partial \theta} > 0$$

Following is the optimal fertility in steady-state $h_H$ :

$$n^*_{s_H} = \left[\frac{1}{(1-\phi+\theta)^{(\alpha_1+2\alpha_2)}(\gamma+\beta\alpha_1)^{(\alpha_1+\alpha_2)}(\beta\alpha_2)^{(\alpha_1-2\alpha_2)}k^{(\alpha_1+\alpha_2)}}\right]^{\frac{1}{\alpha_2}}$$

Derivating $n^*_{s_H}$ with respect to $k$ shows a negative impact of child labor wage on $n^*_{s_H}$ :

$$\frac{\partial n^*_{s_H}}{\partial k} = -\frac{1}{k^{\alpha_1+\alpha_2+1}(\gamma+\beta\alpha_1)^{\alpha_1+\alpha_2}}\left(\beta\alpha_2\right)^{2\alpha_2-\alpha_1}\frac{\alpha_1+\alpha_2}{(\theta-\phi+1)^{\alpha_1+2\alpha_2}}$$
$$\frac{\partial n^*_{s_H}}{\partial k} < 0$$

Derivating $n^*_{s_H}$ with respect to $\theta$ shows a negative impact of minimum skills level on $n^*_{s_H}$ :

$$\frac{\partial n^*_{s_H}}{\partial \theta} = \left[ -\frac{1}{k^{\alpha_1+\alpha_2}(\gamma+\beta\alpha_1)^{\alpha_1+\alpha_2}} \left(\beta\alpha_2\right)^{2\alpha_2-\alpha_1} \frac{\alpha_1+2\alpha_2}{(\theta-\phi+1)^{\alpha_1+2\alpha_2+1}} \right]$$
$$\frac{\partial n^*_{s_H}}{\partial \theta} < 0$$

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